Lossless Calibration Is Stored Memory: A Topological McMillan-Degree and Wigner–Smith Law for Passive Quantum Networks

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Abstract

How many internal passive modes are required to transmit prescribed quantum noise channels without loss while suppressing all signal directions elsewhere? We prove an architecture-independent answer for finite-dimensional rational networks. Let a causal rational inner scattering matrix S have McMillan degree n, signal block G, and complementary loss block C. At distinct regular boundary frequencies ζ_j, suppose that $G(\zeta_j)$ is isometric on a k_j-dimensional input subspace. If G is a strict contraction at one other frequency, then

$$\sum_j k_j \le n.$$

Indeed, every lossless direction lies in $\ker C(\zeta_j)$; a nonzero maximal minor of C therefore has a zero of multiplicity at least k_j. Exterior powers of a minimal Blaschke–Potapov factorization show that no minor can have more than n zeros. The same factorization yields the exact topological delay identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} Q(\theta)\,\mathrm{d}\theta = \deg_{\mathrm{McM}} \mathsf{S} = n, \qquad Q = -\mathrm{i}\mathsf{S}^*\partial_\theta \mathsf{S} \succeq 0.$$

Thus lossless calibration multiplicity is bounded by integrated Wigner–Smith delay and forces peak trace delay at least $\sum_j k_j$. The bound is sharp for every number of ports and calibration nodes. A Rouché version makes the zero count stable under analytic passive perturbations, while an explicit counterexample shows why pointwise approximate calibration alone cannot imply a degree bound. Applied to a previously constructed irreducible six-port reservoir filter, the law counts $3S$ calibration zeros and proves that its degree $3S+6$ is universally optimal up to six states, not merely optimal against reducing comparators. Its determinant gives an exact delay spectrum whose integral equals $3S+6$ and whose peak is exponential. In weak-coupling dephasing, the result turns every collection of exactly preserved spectral directions into a compulsory passive-memory budget. Proofs, passive perturbations, quadrature, root audits and figures are reproduced by the linked public repository.

Keywords: rational inner networks; McMillan degree; Wigner–Smith time delay; passive quantum reservoirs; Blaschke–Potapov factorization; transmission zeros.

1 The question and the answer

Passive filtering is attractive in quantum reservoir engineering because it can reshape a bath without continuous active control. It is nevertheless not resource-free: a sharply selective passive response is stored in internal modes and long dwell times. Previous work established this trade only for a particular scalar all-pass mechanism inside an irreducible multichannel construction [1]. That left the main architectural loophole open: could a completely different passive MIMO realization obey the same lossless calibrations with much less memory?

The answer is no for causal finite-dimensional rational inner networks. The obstruction is topological rather than metric. Exact transmission makes a complementary loss minor vanish. A rational inner transfer of McMillan degree n can supply at most n zeros to any one minor. The winding of its determinant is the same integer n, and the corresponding phase derivative is the trace of the Wigner–Smith lifetime matrix. Consequently

$$\boxed{\text{lossless calibration charge} \;\leq\; \text{McMillan degree} \;=\; \text{integrated Wigner–Smith memory.}} \tag{1}$$

The ingredients surrounding (1) are classical. McMillan degree originated in finite passive-network realization theory [2]; rational lossless matrices admit finite Blaschke–Potapov products [3, 4]; and the lifetime matrix goes back to Wigner and Smith [5, 6]. Tangential interpolation and degree-constrained MIMO interpolation are also mature subjects [7, 8, 9]. Our contribution is the missing implication from *lossless directional calibration of a principal signal block* to a *topological degree and lifetime budget of every passive completion*, including multiplicity, sharpness and a robust analytic certificate. We do not claim a new factorization theorem or a new definition of time delay.

The result advances a resource-boundary programme developed across three earlier manuscripts. The first framed a Heisenberg cut through coherence maintenance costs [10]; the second showed a finite-degree irreducible/reducible MIMO separation [11]; the third embedded that separation in a passive quantum reservoir and exposed a scalar dwell-time price [1]. Here the comparator class disappears: the lower bound applies to every rational inner architecture.

Contributions.

1. A calibration-charge theorem counts independent lossless directions, including several directions at the same frequency.
2. A self-contained exterior-power proof bounds every minor by the full network's McMillan degree.
3. An exact Wigner–Smith sum rule turns the algebraic count into an integrated and peak passive-memory lower bound.
4. A $2m$-port family saturates all inequalities for arbitrary m and any number of calibration frequencies.
5. A Rouché certificate preserves the count under passive analytic perturbations; a degree-one example proves that sampled approximate calibration alone is insufficient.
6. The irreducible reservoir family is re-audited with all $3S$ calibration zeros. Its degree is $3S+6$, within an additive six states of the universal optimum, and its delay integral and peak are computed exactly.

2 Passive transfer model

We use the unit-disc convention. Let $\mathsf{S} : \mathbb{D} \to \mathbb{C}^{N\times N}$ be a rational inner function: it is rational and analytic in $\mathbb{D}$, has no pole at the boundary points under discussion, and satisfies

$$\mathsf{S}(\zeta)^*\mathsf{S}(\zeta) = I_N, \qquad \zeta \in \mathbb{T}. \tag{2}$$

Pure delays z^q are causal in this convention. A minimal state-space realization has dimension $n = \deg_{\mathrm{McM}} \mathsf{S}$. For square rational inner functions this integer also equals the degree, or boundary winding, of the scalar inner function $\det \mathsf{S}$.

Select m signal input columns, p protected output rows and the remaining $\ell = N - p$ loss-output rows. With irrelevant blocks suppressed, write

$$\mathsf{S}(z) = \begin{pmatrix} G(z) & * \\ C(z) & * \end{pmatrix}, \qquad G(z) \in \mathbb{C}^{p\times m}, \quad C(z) \in \mathbb{C}^{\ell\times m}. \tag{3}$$

Column unitarity gives, on every regular boundary point,

$$G(\zeta)^*G(\zeta) + C(\zeta)^*C(\zeta) = I_m. \tag{4}$$

This equality is the only quantum-network identity used in the degree obstruction. It is also the standard preservation law behind passive linear quantum input–output models [12, 13, 14].

Definition 2.1 (Calibration charge)**.** At a regular point $\zeta \in \mathbb{T}$, the lossless calibration space is

$$E_\zeta := \ker(I_m - G(\zeta)^*G(\zeta)), \qquad k_\zeta := \dim E_\zeta.$$

For distinct calibration nodes $\zeta_1, \dots, \zeta_M$, their total charge is $K := \sum_{j=1}^{M} k_{\zeta_j}$.

The definition asks only for norm preservation. It does not require a specified output phase, a common invariant channel, reciprocity or commutativity. From (4),

$$E_\zeta = \ker C(\zeta). \tag{5}$$

A *strict stop* is one regular point ζ_* for which $\|G(\zeta_*)\|_{\mathrm{op}} < 1$. At such a point (4) makes $C(\zeta_*)^*C(\zeta_*)$ positive definite. In particular $\ell \geq m$ and $C(\zeta_*)$ has full column rank.

3 The minor-degree mechanism

Two elementary lemmas carry the proof. We include details because they make the result easy to audit and expose exactly where rationality enters.

Lemma 3.1 (Local nullity consumes determinant multiplicity)**.** *Let $A(z)$ be an $m \times m$ matrix holomorphic near z_0. If $\dim\ker A(z_0) \geq k$, then either $\det A \equiv 0$ or*

$$\operatorname{ord}_{z_0} \det A \geq k. \tag{6}$$

Proof. Choose a constant invertible input change of basis whose first k columns span a subspace of $\ker A(z_0)$. The first k columns of the transformed matrix vanish at z_0, so every entry in those columns is divisible by $z - z_0$. Multilinearity of the determinant supplies at least k factors. Constant basis changes alter the determinant only by a nonzero constant. □

Lemma 3.2 (Every minor inherits the full McMillan budget)**.** *Let S be an $N \times N$ rational inner function of McMillan degree n. Every nonzero $r \times r$ minor of S has at most n finite zeros, counted with multiplicity and restricted to points where the minor is regular.*

Proof. A minimal finite Blaschke–Potapov factorization has exactly n rank-one factors [4]:

$$\mathsf{S}(z) = U \prod_{q=1}^{n} \left[I_N + \big(b_{a_q}(z) - 1\big) u_q u_q^* \right], \quad b_a(z) = \frac{z - a}{1 - \bar{a}z}, quad|a| < 1, \tag{7}$$

where U is constant unitary and $\|u_q\| = 1$. Factors with $a_q = 0$ include integer delays.

Every $r \times r$ minor is an entry of the exterior-power matrix $\bigwedge^r \mathsf{S}$. On $\bigwedge^r \mathbb{C}^N$, a factor in (7) becomes

$$I + \big(b_{a_q}(z) - 1\big) P_q,$$

where P_q is the orthogonal projection onto wedges containing u_q. Thus every entry of the product has a common scalar denominator dividing $\prod_{q=1}^{n}(1 - \bar{a}_q z)$ and a numerator of degree at most n. Cancellation can only lower the degree. A nonzero entry therefore has at most n finite zeros. □

The exterior-power step is essential. Expanding an $r \times r$ determinant entry by entry would give the misleading loose count rn; the compound matrix sees that each rank-one Potapov factor contributes at most one scalar degree to any minor.

4 Universal calibration-charge theorem

Theorem 4.1 (Lossless calibration is stored degree)**.** *Let* S *be a causal* $N \times N$ *rational inner transfer of McMillan degree* n*, partitioned as in* (3)*. Let* $\zeta_1, \dots, \zeta_M \in \mathbb{T}$ *be distinct regular calibration nodes with charges* $k_j = \dim E_{\zeta_j}$*. If there exists a regular strict-stop point* ζ_* *satisfying* $\|G(\zeta_*)\|_{\mathrm{op}} < 1$*, then*

$$K := \sum_{j=1}^{M} k_j \le \deg_{\mathrm{McM}} \mathsf{S} = n. \tag{8}$$

Proof. At the strict stop, $C(\zeta_*)$ has full column rank. Select m of its rows whose square submatrix $C_I(\zeta_*)$ has nonzero determinant, and put $f(z) = \det C_I(z)$. Hence $f \not\equiv 0$.

At node ζ_j, (5) gives $E_{\zeta_j} \subseteq \ker C_I(\zeta_j)$, so $\dim \ker C_I(\zeta_j) \ge k_j$. By Lemma 3.1, f has a zero of multiplicity at least k_j there. The nodes are distinct, whence f has at least K zeros counting multiplicity. It is an $m \times m$ minor of S, so Lemma 3.2 gives $K \le n$. □

Remark 4.2 (Why the strict stop is necessary)*.* If G is isometric everywhere, then $C \equiv 0$ and every loss minor vanishes identically; a constant unitary signal route has degree zero and arbitrarily many lossless "calibrations." One strict stop certifies that the chosen minor is not the zero function. The theorem is therefore a selectivity law, not a charge for transparent wiring.

Remark 4.3 (No hidden architecture comparator)*.* The proof permits arbitrary frequency-dependent channel mixing, arbitrary constant unitary pre- and post-processing, nonreciprocal inner networks, and any number $\ell \ge m$ of loss outputs. It compares neither with a reducing class nor with a fixed numerator denominator bidegree. This is the precise sense in which Theorem 4.1 closes the main loophole left by the earlier architecture separation [1].

5 Topological Wigner–Smith memory law

For $\zeta = \mathrm{e}^{\mathrm{i}\theta}$ define the dimensionless Wigner–Smith matrix

$$Q(\theta) := -\mathrm{i}\, \mathsf{S}(\mathrm{e}^{\mathrm{i}\theta})^* \frac{\mathrm{d}}{\mathrm{d}\theta} \mathsf{S}(\mathrm{e}^{\mathrm{i}\theta}). \tag{9}$$

In a physical frequency variable ω, multiplication by $\mathrm{d}\theta/\mathrm{d}\omega$ restores units of time. The sign in (9) matches the analytic-disc convention; changing to a z^{-1} engineering convention reverses the orientation and requires the corresponding sign change.

Theorem 5.1 (Degree is Wigner–Smith phase volume)**.** *For a causal rational inner* S *of McMillan degree* n*,*

$$Q(\theta) \succeq 0, \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta = \operatorname{wind}_{\mathbb{T}} \det \mathsf{S} = n. \tag{10}$$

Proof. For a single factor $E_q = I + (b_{a_q} - 1) u_q u_q^*$ in (7), direct differentiation gives a rank-one positive delay

$$Q_{E_q}(\theta) = \frac{1 - |a_q|^2}{|\mathrm{e}^{\mathrm{i}\theta} - a_q|^2}\, u_q u_q^*, \tag{11}$$

whose trace integrates to 2π. If A and B are inner, the product rule is

$$Q_{AB} = B^* Q_A B + Q_B.$$

Positivity therefore survives the entire Potapov product, and trace invariance under unitary conjugation makes the integrated traces add to $2\pi n$.

Equivalently, cyclicity of trace gives $\operatorname{tr} Q = -\mathrm{i}\, \partial_\theta \log \det \mathsf{S}$. Each rank-one factor has determinant b_{a_q}, so $\det \mathsf{S}$ winds n times. □

Combining the two theorems yields the operational statement.

Corollary 5.2 (Universal memory budget). *Under the hypotheses of Theorem 4.1,*

$$K \le n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} Q(\theta)\,\mathrm{d}\theta \le \sup_\theta \operatorname{tr} Q(\theta). \tag{12}$$

Moreover,

$$\sup_\theta \lambda_{\max}(Q(\theta)) \ge \frac{K}{N}. \tag{13}$$

Consequently a uniform proper-delay cap $Q(\theta) \preceq D I_N$ permits at most $K \le ND$ units of lossless calibration charge.

The lower bound concerns phase volume, not only a single resonant peak. A network may distribute the required memory broadly over frequency or concentrate it into sharp long-lived modes, but it cannot remove the integral. This distinction complements the earlier scalar stopband theorem, which forced an exponentially high peak for one particular suppression mechanism [1].

6 Sharpness at every dimension

The coefficient one in (8) cannot be improved.

Proposition 6.1 (Exact saturation). *For every $m, M \ge 1$ there is a causal $2m$-port rational inner transfer with M distinct nodes, charge m at each node, a perfect stop, and*

$$\deg_{\mathrm{McM}} \mathsf{S} = mM = \sum_{j=1}^{M} k_j. \tag{14}$$

Its Wigner–Smith trace is the constant mM, so the integrated bound and the peak-trace bound are also equalities.

Proof. Let

$$H_m = \frac{1}{\sqrt{2}}\begin{pmatrix} I_m & I_m \\ I_m & -I_m \end{pmatrix}, \qquad \mathsf{S}_{m,M}(z) = H_m \operatorname{diag}(I_m, z^M I_m) H_m.$$

Then

$$G(z) = \frac{1+z^M}{2} I_m, \qquad C(z) = \frac{1-z^M}{2} I_m. \tag{15}$$

At every Mth root of unity, $C = 0$ and $k_j = m$. At every solution of $z^M = -1$, $G = 0$. Since $\det \mathsf{S} = z^{mM}$, its McMillan degree is mM. Constant conjugation of $\operatorname{diag}(0, M I_m)$ gives Q, so $\operatorname{tr} Q = mM$. □

This saturating family is reducible and deliberately simple. It proves that irreducibility is not needed for the universal resource law. Irreducibility becomes useful for attaining strong stopband performance under richer directional calibrations, as revisited in Section 8.

7 What robustness is possible—and what is not

Exact boundary equalities are experimentally idealized. Two different notions of robustness must be separated.

7.1 Analytic zero-count stability

Let f_0 be the nonzero loss minor selected in the proof of Theorem 4.1. Choose pairwise disjoint positively oriented Jordan contours $\Gamma_1, \ldots, \Gamma_M$ in the complex frequency plane, avoiding all poles, so that the interior of Γ_j contains the calibration zero ζ_j with multiplicity at least k_j.

Theorem 7.1 (Rouché-stable calibration charge)**.** *Let $\widetilde{\mathsf{S}}$ be another rational inner transfer, regular on and inside every Γ_j, and let $\widetilde{f}$ denote the same loss minor. If*

$$\sup_{z\in\Gamma_j} |\widetilde{f}(z) - f_0(z)| < \inf_{z\in\Gamma_j} |f_0(z)| \qquad (j = 1, \ldots, M), \tag{16}$$

then

$$\deg_{\mathrm{McM}} \widetilde{\mathsf{S}} \geq \sum_{j=1}^{M} k_j. \tag{17}$$

More generally the right side may be replaced by the total number of zeros of f_0 inside the contours, counted with multiplicity.

Proof. Rouché's theorem [15] gives the same zero count for f_0 and $\widetilde{f}$ inside each contour. Their interiors are disjoint, so $\widetilde{f}$ has at least $\sum_j k_j$ zeros. Apply Lemma 3.2 to $\widetilde{\mathsf{S}}$. ☐

This is a robust *model certificate*: it uses a rational fit or analytic component model on small complex-frequency contours. It does not pretend that off-axis transfer values are direct laboratory observables. In practice, one fits a passive rational model to real-frequency scattering data, propagates parameter uncertainty to (16), and verifies the zero count on the model family.

7.2 Why pointwise approximate calibration is insufficient

Proposition 7.2 (No count from unseparated approximate samples)**.** *Fix any integer M and tolerance $\delta > 0$. There is a degree-one two-port rational inner transfer, M distinct boundary nodes, and a perfect stop such that the signal magnitude at every node is at least $1 - \delta$.*

Proof. For $0 < r < 1$ set

$$b_r(z) = \frac{z - r}{1 - rz}, \qquad \mathsf{S}_r(z) = H_1 \operatorname{diag}(1, b_r(z)) H_1, \qquad G_r(z) = \frac{1 + b_r(z)}{2}.$$

This network has degree one and $G_r(-1) = 0$. Continuity and $G_r(1) = 1$ give an open boundary arc around 1 on which $|G_r| \geq 1 - \delta$. Select any M distinct nodes in that arc. ☐

Thus a theorem depending only on the number of approximate samples is false: the nodes may oversample one degree-one pass window. Valid finite-tolerance bounds need additional information—for example node separation plus a derivative budget, or the analytic contour margin in Theorem 7.1. This no-go prevents a common but serious overinterpretation of exact root counts.

8 Near-optimal irreducible quantum reservoir filter

We now apply the universal law to the explicit six-port family of [1]. The purpose is not to repeat its architectural separation, but to answer a question that paper could not: how far is its internal dimension from the best *arbitrary* passive architecture?

Fix $\alpha = 9/20$ and $S \geq 5$, and put

$$r_S = 1 - \mathrm{e}^{-\alpha S}, \quad b_r(z) = \frac{z - r}{1 - rz}, \quad B_S(z) = \eta_S b_{r_S}(z)^S, \tag{18}$$

where $\eta_S = 1$ for odd S and -1 for even S. Hence $B_S(-1) = -1$. Let $\phi_g = (g-1)(1-r_S)$ for $g = 0, 1, 2$ and $B_g = e^{i\phi_g} B_S$. With the 3×3 Fourier matrix F, define

$$U(z) = F \operatorname{diag}(1, z, z^2) F^*, \qquad U_{\mathrm{rev}}(z) = F \operatorname{diag}(z^2, z, 1) F^*. \tag{19}$$

The balanced interferometers and the full network are

$$W_g(z) = \frac{1}{2}\begin{pmatrix} 1 + B_g(z) & 1 - B_g(z) \\ 1 - B_g(z) & 1 + B_g(z) \end{pmatrix}, \qquad \mathsf{S}_S = \operatorname{diag}(U, I_3)\, W \, \operatorname{diag}(U_{\mathrm{rev}}, I_3), \tag{20}$$

where W places the three W_g on signal/loss port pairs. Every factor is inner. The signal and loss blocks from the three signal inputs are

$$G_S(z) = U(z) \operatorname{diag}\left(\frac{1 + B_0(z)}{2}, \frac{1 + B_1(z)}{2}, \frac{1 + B_2(z)}{2} \right) U_{\mathrm{rev}}(z), \tag{21}$$

$$C_S(z) = \operatorname{diag}\left(\frac{1 - B_0(z)}{2}, \frac{1 - B_1(z)}{2}, \frac{1 - B_2(z)}{2} \right) U_{\mathrm{rev}}(z). \tag{22}$$

Each scalar degree-S inner function B_g takes the value one at exactly S simple points of $\mathbb{T}$. The three root sets are disjoint because the phases ϕ_g are distinct modulo 2π. At a root z of $B_g - 1$, the input direction $v_{g,z} = U(z) e_g$ is lossless and obeys $G_S(z) v_{g,z} = z^2 v_{g,z}$. Hence the total calibration charge is

$$K_S = 3S. \tag{23}$$

At $z = -1$ every $|1 + B_g(-1)|/2 < 1$, so the signal block is strictly contractive. The universal theorem already forces degree at least $3S$.

The realized degree is exact, not an entrywise upper bound. Since $\det U = \det U_{\mathrm{rev}} = z^3$ and $\det W_g = B_g$,

$$\det \mathsf{S}_S(z) = z^6 \prod_{g=0}^{2} B_g(z), \qquad \det C_S(z) = z^3 \prod_{g=0}^{2} \frac{1 - B_g(z)}{2}. \tag{24}$$

Therefore

$$\deg_{\mathrm{McM}} \mathsf{S}_S = 3S + 6 = K_S + 6. \tag{25}$$

The six-state overhead is exactly the two frequency-dependent mixers. Thus the same device previously proved superior to every calibrated reducing comparator is now certified within an additive six states of the best possible passive network of *any* architecture.

The determinant also gives the complete trace delay:

$$\operatorname{tr} Q_S(\theta) = 6 + 3S \frac{1 - r_S^2}{1 - 2 r_S \cos\theta + r_S^2}. \tag{26}$$

Consequently

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr} Q_S \, d\theta = 3S + 6, \qquad \max_{\theta} \operatorname{tr} Q_S(\theta) = 6 + 3S \frac{1 + r_S}{1 - r_S} = 6 + 3S(2e^{\alpha S} - 1). \tag{27}$$

The area is the compulsory topological memory. The exponentially high peak is the additional price of concentrating the scalar all-pass phases sharply enough to achieve exponential stop suppression.

9 From scattering memory to decoherence selectivity

The theorem is a network constraint; its quantum consequence follows through the standard weak-coupling spectral interface. Let a positive bath spectral matrix obey

$$j_- I_m \preceq J(\omega) \preceq j_+ I_m, \qquad 0 < j_- \le j_+ < \infty, \tag{28}$$

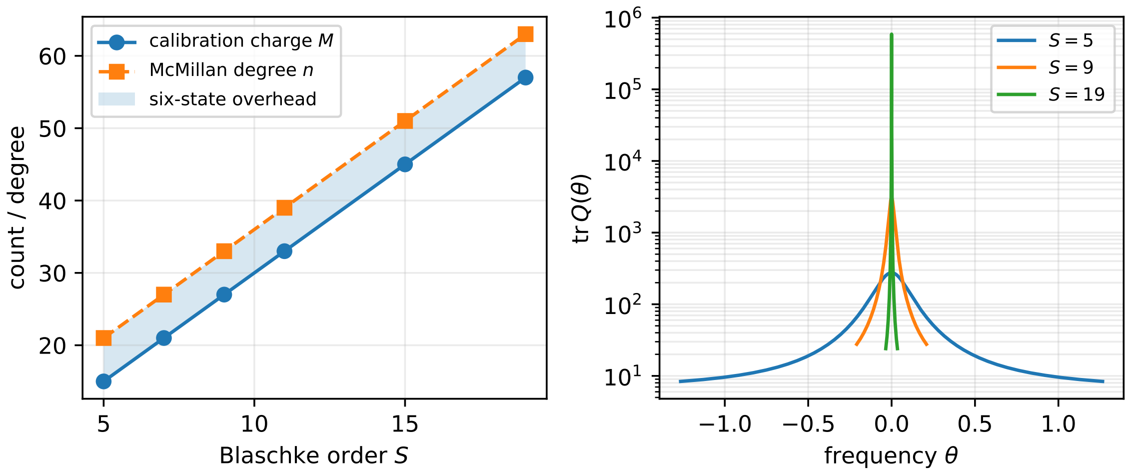

Figure 1: Left: the explicit irreducible six-port construction has calibration charge $K_S = 3S$ and exact McMillan degree $n_S = 3S + 6$, a constant six-state overhead above the universal lower bound. Right: the same fixed phase volume becomes increasingly concentrated into a high Wigner–Smith peak as r_S approaches the unit circle. The frequency windows differ by S so the peak shapes remain visible.

and define the signal contribution to a Kossakowski matrix by

$$\Gamma_G(\omega) = G(\mathrm{e}^{\mathrm{i}\theta(\omega)})J(\omega)G(\mathrm{e}^{\mathrm{i}\theta(\omega)})^*. \tag{29}$$

This congruence is the frequency-domain form appearing in weak-coupling Markov generators [16, 17].

At a calibration where $Gv = \lambda v$ with $|\lambda| = 1$, contraction implies $G^*v = \bar{\lambda}v$. Hence

$$v^*\Gamma_G v = v^*Jv \geq j_-. \tag{30}$$

At a stop where $\|G\|_{\mathrm{op}} \leq \varepsilon$,

$$\|\Gamma_G\|_{\mathrm{op}} \leq j_+\varepsilon^2. \tag{31}$$

The spectral-rate contrast is therefore squared at the Kossakowski level. The new statement is the compulsory resource behind preserving many pass directions:

$$\#\{\text{independent exactly preserved spectral directions}\} \leq \deg_{\mathrm{McM}} \mathsf{S} = \frac{1}{2\pi}\int \mathrm{tr}\, Q. \tag{32}$$

Auxiliary vacuum ports remain physical channels. They can add a baseline to an observed Ramsey decay, as accounted for explicitly in [1]; Theorem 4.1 neither removes nor hides that floor. What it proves is orthogonal: regardless of how the passive network is realized, exact preservation of K selected spectral directions together with any strict all-direction stop requires at least K internal-state degrees and the same amount of integrated delay.

This is also the precise connection to the Heisenberg-cut resource viewpoint [10]. Active maintenance can pay in work and entropy production; a passive architecture can avoid that continuous drive, but its selective boundary conditions are stored as dynamical memory. No universal conversion from McMillan degree to joules is asserted: such a conversion depends on linewidths, temperature, fabrication and initialization. The universal statement is the topological state/delay count.

10 Reproducible certificate and adversarial checks

The proof of Theorems 4.1 and 5.1 is analytic. Computation audits the explicit family and deliberately probes the fragile parts of the argument. The public repository [18] runs the following checks.

1. It generates all $3S$ closed-form calibration nodes for $S \in \{5, 7, 9, 11, 15, 19\}$ and evaluates $\det C_S$ there.

2. It verifies the exact determinant degrees $3S + 6$ and $3S + 3$ for the full network and selected loss minor.

3. Adaptive quadrature checks $(2\pi)^{-1} \int \operatorname{tr} Q_S = 3S + 6$, including the narrow $S = 19$ resonance.

4. It checks strict nontriviality of the loss minor at the stop $z = -1$.

5. At $S = 5$, it perturbs the input and output beam-splitter angles while preserving passivity. Fifteen disjoint contours are constructed around the reference calibration zeros. The sampled maximum Rouché ratio is below one, and an independent polynomial root count finds one perturbed zero in every contour.

6. A fail-closed verifier rejects a missing record, incorrect degree, failed sum rule, trivial stop minor, non-strict Rouché margin or escaped root.

7. As an adversarial check of the exterior-power step, 128 seeded random six-port Potapov products test minors of every size from one through six and factor counts up to nine. After multiplication by the single product denominator, every minor is fitted within the predicted polynomial degree budget and checked on an independent complex grid.

S	charge K_S	degree n_S	$\frac{1}{2\pi} \int \operatorname{tr} Q_S$	$\max \operatorname{tr} Q_S$	$\lvert \det C_S(-1) \rvert$
5	15	21	21.0000000000	2.7563×10^2	0.9972253
9	27	33	33.0000000000	3.0785×10^3	0.9999241
19	57	63	62.9999999909	5.8896×10^5	0.99999999

Table 1: Selected rows from the machine-readable certificate. The small $S = 19$ quadrature discrepancy is below 10^{-8} relative error despite the exponentially narrow delay peak.

For the passive perturbation audit, the beam-splitter rotation is $R(\eta) H \operatorname{diag}(1, B_g) H R(\xi)$ in each signal/loss pair. The run uses angular perturbation 2×10^{-4} radians. Across the fifteen contours the largest sampled ratio

$$\frac{\max_{\Gamma_j} |\widetilde{f} - f_0|}{\min_{\Gamma_j} |f_0|} \tag{33}$$

is 0.04512; every independently computed perturbed numerator has exactly one root in each corresponding contour. This numerical audit illustrates Theorem 7.1; the theorem itself does not depend on grid sampling.

The complete replay is:

```
python research/calibration_memory_certificate.py
python verification/verify_calibration_memory.py
```

It emits `results/calibration_memory/certificate.json` and regenerates Fig. 1. The JSON file records tolerances and unrounded values. The random-minor campaign does not replace Lemma 3.2; it is a falsification attempt against coding mistakes and an accidental rn-rather-than-n denominator count.

11 Relation to prior theories and novelty boundary

It is useful to separate the present statement from nearby classical results.

Tangential interpolation. General and bitangential interpolation theories ask for a rational matrix matching prescribed directional values, often with complexity constraints [7, 9]. Model-reduction interpolation matches moments or tangential transfer data [8]. Those frameworks do not by themselves identify the complementary loss block of a unitary completion, turn exact norm saturation into transmission-zero multiplicity, and equate the resulting lower bound with lifetime phase volume.

Blaschke–Potapov factorization. Potapov products describe rational inner matrices and expose their degree [3, 4]. They are used in passive quantum networks to separate trapped modes from propagation delays [19]. We use that theorem, through exterior powers, as a sharp budget for *every loss minor*; the factorization itself is not new.

Wigner–Smith theory. The lifetime matrix and total scattering phase are classical [5, 6]. The new content is not the trace-log-determinant identity. It is the lower bound on that trace integral from directional lossless calibrations of a principal block, plus its sharp realization and passive-network consequence.

Earlier architecture separation. The previous paper proved that constant-channel reducing comparators cannot match an irreducible construction at fixed bidegree, and obtained a scalar peak-delay price [1]. The current theorem neither assumes a reducing comparator nor uses full-spark signatures. It certifies a minimum full-network realization degree for every passive architecture. The two results are complementary: one prices architectural simplification; the other prices lossless selectivity itself.

To the best of our knowledge after a targeted search of these literatures, the combined calibration-charge inequality (8) and its Wigner–Smith form (12) have not been stated in this passive signal/loss setting. This is a scoped novelty claim, not a claim that root counting, compound matrices, inner factorization or phase-delay sum rules are individually new.

12 Limitations and escape routes

Limitation 12.1 (Finite rational memory)*.* The theorem excludes singular-inner factors, ideal incommensurate distributed delays and genuinely infinite-dimensional continua. A finite rational approximation has a finite McMillan degree and is covered; the convergence of such approximations is a separate problem.

Limitation 12.2 (Exact versus robust calibration)*.* Exact point calibration gives the clean topological charge. Robustness is available when an analytic contour margin is certified. Pointwise approximate samples alone do not suffice, by Theorem 7.2. Deriving optimal noisy-data bounds from separated real-frequency samples and a measured delay cap remains open.

Limitation 12.3 (Passivity)*.* Active gain, time variation and measurement-based feedback fall outside rational inner scattering. Such resources may reduce passive memory, but they reintroduce pumping, control or entropy costs rather than refuting the passive theorem.

Limitation 12.4 (Degree is not energy)*.* McMillan degree and Wigner–Smith delay are architecture-independent memory coordinates. They do not fix fabrication volume, stored energy or thermodynamic work without a component model. The paper therefore does not derive a numerical Landauer cost from (8).

Limitation 12.5 (Markov interface)*.* The decoherence interpretation uses a weak-coupling spectral generator. Strong coupling, non-Markovian initial correlations and driven nonequilibrium baths need a larger dynamical model. The scattering theorem itself remains a deterministic statement about the passive transfer.

These limitations suggest concrete next questions rather than cosmetic extensions: a real-axis noisy-data analogue with optimal separation dependence; an infinite-dimensional version formulated through spectral shift; and a component-level lower bound converting delay phase volume into stored energy or physical volume under linewidth constraints.

13 Conclusion

Perfect selective transmission cannot be specified for free in a passive finite-dimensional network. Each independent lossless direction forces a zero of the complementary loss block; local nullity counts its multiplicity; and a single strict stop selects a nonzero minor whose zero budget cannot exceed the full McMillan degree. The same integer is the winding of the scattering determinant and the integrated Wigner–Smith lifetime. Hence lossless calibration is stored memory.

The inequality is coefficient-sharp, robust under an explicit analytic Rouché margin, and false under unstructured approximate sampling—all three facts matter for a referee-proof statement. For the irreducible six-port reservoir construction, it upgrades an architecture-relative separation into a universal near-optimality theorem: $3S$ calibrated directions require at least $3S$ states, while the construction uses $3S + 6$. Its exponential decoherence selectivity is therefore not achieved by hiding complexity in an unpriced channel transformation. The unavoidable memory is visible both as topological degree and as Wigner–Smith delay.

A Multiplicity at a matrix zero

We record a coordinate-free variant of Lemma 3.1. Let $A(z) : X \to Y$ be an analytic family between two m-dimensional complex spaces. If $\dim \ker A(z_0) = k$, select a decomposition $X = E \oplus E^\perp$ with $E \subseteq \ker A(z_0)$ and $\dim E = k$. In adapted bases,

$$A(z) = \begin{pmatrix} (z - z_0)A_{11}(z) & A_{12}(z) \\ (z - z_0)A_{21}(z) & A_{22}(z) \end{pmatrix},$$

where the first block column has width k. Every term in the Leibniz formula for $\det A$ selects one entry from each of those columns and hence contains $(z - z_0)^k$. This proves $\operatorname{ord}_{z_0} \det A \geq k$ even when the rank defect is nongeneric. Equality holds for a transversal zero; a higher-order degeneracy only strengthens Theorem 4.1.

For the rectangular loss matrix $C : \mathbb{C}^m \to \mathbb{C}^\ell$, the strict stop gives a set I of m output rows with $C_I(\zeta_*)$ invertible. At every calibration node, $E_\zeta \subseteq \ker C \subseteq \ker C_I$, so the same argument applies to this single minor at all nodes. No frequency-dependent choice of rows is made.

B Exterior powers and the degree-one contribution

For completeness, if $E(z) = I + (b(z) - 1)P$ with $P = uu^*$ rank one, then $\mathbb{C}^N = \operatorname{span}\{u\} \oplus u^\perp$. Its rth exterior power splits as

$$\bigwedge^r \mathbb{C}^N = \left(u \wedge \bigwedge^{r-1} u^\perp \right) \oplus \bigwedge^r u^\perp.$$

On the first summand $\bigwedge^r E$ acts by $b(z)$; on the second it acts by one. Hence

$$\bigwedge^r E(z) = I + (b(z) - 1)\Pi$$

for a constant projection Π. Its entries have scalar numerator and denominator degree at most one. A product of n such matrices has a common denominator of degree at most n and entry numerators of degree at most n. The coordinates of $\bigwedge^r \mathsf{S}$ in the standard wedge basis are exactly the signed $r \times r$ minors of S.

This also explains why the degree of a square inner matrix equals the degree of its determinant. For $r = N$, the exterior space is one-dimensional and each rank-one factor contributes precisely b_{a_q}:

$$\det \mathsf{S}(z) = \det U \prod_{q=1}^{n} b_{a_q}(z).$$

C Certificate schema and numerical conditioning

The machine-readable ledger stores, for every S, the calibration multiplicity, full-network degree, loss-minor degree, loss-zero residual, strict-stop determinant, quadrature result, quadrature error estimate and peak trace delay. Closed-form phase inversion is used for calibration nodes; high-degree polynomial root finding is intentionally avoided in the scaling table because clustered roots near the unit circle are ill-conditioned.

Polynomial roots are used only in the $S = 5$ perturbation audit, where they serve as an independent check on the contour calculation. The reference contours have minimum radius 2.11×10^{-4}; the smallest sampled $|f_0|$ is 2.45×10^{-5} and the largest sampled perturbation is 1.79×10^{-6}. The global maximum of the per-contour ratio is smaller than the ratio of these two global extrema and is recorded as 0.04512. The JSON values, rather than the rounded manuscript values, are normative for the reproducibility test. The same ledger records the random Potapov seed, all tested compound ranks and the worst independent-grid interpolation residual.

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