AIRR.SCIENCE · ARR-2026-0744Q5DQQJ8AV8QC v1 [Combinatorics] 28 Nov 2026

THE 2-PEBBLING PROPERTY, SHARP t-PEBBLING BOUNDS, AND GRAHAM'S CONJECTURE FOR SPLIT GRAPHS

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Abstract. A connected graph G satisfies Chung's 2-pebbling property (2PP) if, for every target vertex $r \in V(G)$ and every pebble configuration D satisfying $p(D) + q(D) > 2\pi(G)$ (where $p(D)$ is the total number of pebbles and $q(D)$ is the number of occupied vertices), two pebbles can be moved to r. Despite the structural simplicity of split graphs—chordal graphs of diameter at most 3 whose pebbling numbers were characterized by Alcón, Gutierrez, and Hurlbert (2014)—the validity of 2PP, the tight behavior of the t-pebbling number $\pi_t(G)$, and Graham's product conjecture $\pi(G \,\square\, H) \le \pi(G)\pi(H)$ for split graphs have remained open.

We resolve all three questions completely. Our central structural tool is an *Intra-Clique Move Elimination Lemma*: any minimum-length pebbling sequence delivering t pebbles to a target r in a split graph $G = (C \cup I, E)$ avoids all moves entering $I \setminus \{r\}$, all moves inside $C \setminus \{r\}$ when $r \in C$, and all moves $C_1 \to C$ and $C_0 \to C_0$ when $r \in I$, thereby compressing optimal pebbling into a layered directed acyclic graph of path length at most 3 (and at most 2 when $r \in C$). Combining this compression with causal binary-tree extraction yields the sharp bound $\pi_t(G) \le \pi(G) + 2^{\mathrm{diam}(G)}(t-1)$ for all connected split graphs and all $t \ge 1$, attained across every diameter $d \in \{0,1,2,3\}$ in both non-Class-0 ($\delta = 1$) and Class-0 ($\delta = 3$) regimes, while exhibiting a strict sub-maximal frontier $\pi_t(G_{4,4}) = 8t < \pi_1(G_{4,4}) + 8(t-1)$ for $t \ge 2$. Coupling the same DAG compression with an upward even-to-odd parity boost on $I \setminus \{r\}$, a conservation identity, and an optimal parity-absorption routing analysis proves that every connected split graph satisfies 2PP (and hence O2PP; in particular, no split graph is a Lemke graph). Finally, we establish Graham's conjecture whenever G is a connected split graph and H is any tree, complete graph, hypercube Q_d, complete bipartite graph, or complete multipartite graph, including a self-contained lifting lemma for the odd 2-pebbling property that bypasses strict product drops such as $\pi((K_3 + \text{leaf}) \,\square\, K_2) = 9 < 10 = 2\pi(K_3 + \text{leaf})$.

1. Introduction

Graph pebbling, introduced by Fan Chung [2] following a number-theoretic question of Lagarias and Saks (see also Lemke and Kleitman [9]), models resource transport across a connected finite simple graph $G = (V, E)$. Given an initial pebble configuration $D\colon V \to \mathbb{Z}_{\ge 0}$, a *pebbling move* $(u \to v)$ along an edge $uv \in E$ removes two pebbles from u and places one pebble on v. For a target vertex $r \in V$ and integer $t \ge 1$, a configuration D is *t-solvable for r* if there exists a finite sequence of valid pebbling moves transforming D into a configuration D' with $D'(r) \ge t$. The *t-pebbling number* $\pi_t(G, r)$ is the smallest integer m such that every configuration D of size $p(D) := \sum_{v \in V} D(v) \ge m$ is t-solvable for r, and $\pi_t(G) := \max_{r \in V} \pi_t(G, r)$. When $t = 1$, we write $\pi(G) = \pi_1(G)$ for the classical pebbling number. Clearly $\pi(G) \ge \max\{n, 2^{\mathrm{diam}(G)}\}$ for every connected graph on $n = |V|$ vertices; graphs satisfying $\pi(G) = n$ are called *Class* 0.

To attack product bounds via induction, Chung [2] introduced a fundamental strengthening of $\pi(G)$ based on the support size $q(D) := |\{v \in V : D(v) \ge 1\}|$. A connected graph G satisfies the *2-pebbling property* (2PP) if, for every target $r \in V$ and every configuration D satisfying

$$p(D) + q(D) > 2\pi(G), \tag{1}$$

D is 2-solvable for r. Graphs that fail (1) are known as *Lemke graphs*. A closely related weakening tracks the number of vertices with an odd number of pebbles, $o(D) := |\{v \in V : D(v) \equiv 1 \pmod 2\}|$: a graph satisfies the *odd 2-pebbling property* (O2PP) if $p(D) + o(D) > 2\pi(G)$ guarantees 2-solvability for every target $r \in V$. Since $o(D) \le q(D)$ pointwise, 2PP implies O2PP. Both properties lie at the heart of *Graham's pebbling conjecture* [2], which asserts that for all connected graphs G and H,

$$\pi(G \,\square\, H) \le \pi(G)\pi(H), \tag{2}$$

where $G \,\square\, H$ denotes the Cartesian product. Over the past three decades, 2PP has been proved to force (2) whenever H is a complete graph or hypercube under tight lifting [2], a tree [10, 11], a complete bipartite graph [4], or a complete multipartite graph [12]; see also the surveys of Hurlbert [7, 8].

2020 *Mathematics Subject Classification.* Primary 05C57; Secondary 05C35, 05C76.

Key words and phrases. Graph pebbling, split graphs, 2-pebbling property, odd 2-pebbling property, t-pebbling, Graham's conjecture, Lemke graphs.

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A graph G is a *split graph* if $V(G)$ admits a partition $V(G) = C \sqcup I$ into a clique C and an independent set I. Split graphs form a central subclass of chordal graphs, are recognizable in linear time from their degree sequences [5], and satisfy $\mathrm{diam}(G) \le 3$. In a landmark paper, Alcón, Gutierrez, and Hurlbert [1] derived a closed-form formula and an $O(n^\beta)$ algorithm ($\beta < 1.41$) for $\pi(G)$ on split graphs, and proved that every connected split graph with minimum degree $\delta(G) \ge 3$ is Class 0. Nevertheless, three natural questions left open since [1] and the product literature [2, 12] remained unresolved: whether split graphs can ever produce Lemke graphs (i.e., fail 2PP), whether their t-pebbling numbers obey the sharp diameter scaling $\pi_t(G) \le \pi(G) + 2^{\mathrm{diam}(G)}(t-1)$, and whether split graphs satisfy Graham's conjecture against the standard 2PP product families.

In this paper, we answer all three questions affirmatively with completely self-contained combinatorial proofs.

Theorem 1.1 (Full 2-Pebbling Property for Split Graphs)**.** *Every connected split graph G satisfies the 2-pebbling property (2PP): for every target $r \in V(G)$ and every configuration D with $p(D) + q(D) > 2\pi(G)$, two pebbles can be moved to r. Consequently, no split graph is a Lemke graph, and every connected split graph satisfies the odd 2-pebbling property (O2PP).*

Theorem 1.2 (Sharp t-Pebbling Bound for Split Graphs)**.** *For every connected split graph G and every integer $t \ge 1$,*

$$\pi_t(G) \le \pi(G) + 2^{\mathrm{diam}(G)}(t-1). \tag{3}$$

Moreover, equality holds for every diameter $d \in \{0,1,2,3\}$ and every $t \ge 1$; in diameter 3, equality is attained both by the infinite non-Class-0 ($\delta = 1$) family $K_m + \{r,u\}$ ($m \ge 2$, $\pi_t = n + 8t - 4$) and by the Class-0 ($\delta = 3$) graph G_8 ($\pi_t(G_8) = 8t$). By contrast, the Class-0 ($\delta = 4$) diameter-3 split graph $G_{4,4}$ exhibits a strict sub-maximal jump $\pi_1(G_{4,4}) = 10$ while $\pi_t(G_{4,4}) = 8t < \pi_1(G_{4,4}) + 8(t-1)$ for all $t \ge 2$.

Theorem 1.3 (Graham's Pebbling Conjecture for Split Graphs)**.** *Let G be any connected split graph, and let H be any tree, complete graph K_m, hypercube Q_d, complete bipartite graph, or complete multipartite graph. Then*

$$\pi(G \,\square\, H) \le \pi(G)\pi(H).$$

A subtle point arises in the hypercube case $H = Q_d$ ($d \ge 2$) of Theorem 1.3. Chung's classical K_2-product theorem [2, Theorem 5] asserts that if G satisfies 2PP, then $\pi(G \,\square\, K_2) \le 2\pi(G)$, and *if equality* $\pi(G \,\square\, K_2) = 2\pi(G)$ *holds*, then $G \,\square\, K_2$ again satisfies 2PP. However, strict drops $\pi(G \,\square\, K_2) < 2\pi(G)$ genuinely occur among connected split graphs: as we prove and verify computationally in Section 5, the diameter-2 split graph $K_3 + \text{leaf}$ has $\pi = 5$ yet $\pi((K_3 + \text{leaf}) \,\square\, K_2) = 9 < 10$. To keep Theorem 1.3 completely rigorous for all Q_d ($d \ge 1$), we prove a clean *Cube Lifting Lemma* (Lemma 5.1) showing that the relaxed threshold property O2PP relative to an upper bound $M \ge \max\{\pi(G), |V(G)|\}$ propagates unconditionally to $(G \,\square\, K_2, 2M)$ because removing pairs from vertices preserves their pebble parity.

The remainder of the paper is organized as follows. Section 2 establishes the Intra-Clique Move Elimination Lemma, the upward even-to-odd boost on $I \setminus \{r\}$, and our two structural lower bounds (Lemmas 2.3 and 2.4). Section 3 proves Theorem 1.2 via causal binary-tree extraction and establishes the exact extremal and frontier properties of $K_m + \{r,u\}$, G_8, and $G_{4,4}$. Section 4 presents the complete proof of Theorem 1.1 via conservation identities and parity-absorption routing. Section 5 proves Lemma 5.1 and Theorem 1.3. Section 6 summarizes independent exhaustive computational checks across all 164 connected split graphs on $n \le 7$ vertices and closes with open problems.

2. Structural Compression and Lower-Bound Lemmas

Throughout, $G = (V, E)$ denotes a connected split graph on $n = |V|$ vertices with split partition $V = C \sqcup I$, where C is a clique and I is an independent set. Whenever G is non-complete ($G \not\cong K_n$), we choose C to be a *maximal* clique; then $k := |C| \ge 2$, $I \ne \emptyset$, and every $u \in I$ satisfies $1 \le |N(u)| \le k-1$ (since G is connected and $C \cup \{u\}$ is not a clique). When $r \in I$, we partition C into

$$C_1 := N(r) \subseteq C \qquad \text{and} \qquad C_0 := C \setminus C_1,$$

so $C_1 \ne \emptyset$, and we partition $I \setminus \{r\}$ according to distance to r:

$$I_2 := \{w \in I \setminus \{r\} : N(w) \cap C_1 \ne \emptyset\}, \qquad I_3 := \{u \in I \setminus \{r\} : N(u) \subseteq C_0\}.$$

Clearly $\mathrm{dist}(w, r) = 2$ for $w \in I_2$ and $\mathrm{dist}(u, r) = 3$ for $u \in I_3$.

2.1. Intra-Clique Move Elimination.

A cornerstone of our analysis is that minimum-length pebbling sequences on split graphs admit a rigid layered DAG compression.

Lemma 2.1 (Intra-Clique Move Elimination Lemma)**.** *Let $G = (C \sqcup I, E)$ be a connected split graph, let $r \in V(G)$, let $t \ge 1$, and let $\sigma = (m_1, \ldots, m_\ell)$ be a minimum-length valid pebbling sequence from a configuration D that places at least t pebbles on r. Then:*

(0) *No move of σ leaves r.*
(1) *No move of σ enters $I \setminus \{r\}$.*
(2) *If $r \in C$, no move of σ has both endpoints in $C \setminus \{r\}$; if $r \in I$, no move of σ goes from C_1 to C.*
(3) *If $r \in I$, no move of σ has both endpoints in C_0.*

Consequently, σ uses only edges of a layered directed acyclic graph $\vec{H}(G, r)$ whose directed paths have length at most 2 when $r \in C$ (with layers $I \setminus \{r\} \to C \setminus \{r\} \to r$, plus direct edges $I \cap N(r) \to r$) and length at most 3 when $r \in I$ (with layers $I_3 \cup I_2 \to C_0 \to C_1 \to r$, plus direct edges $I_2 \to C_1$).

Proof. Let $D_0 := D$ and let D_s denote the configuration immediately after move m_s ($1 \le s \le \ell$). Notice that if r and D allow a valid sequence of length ℓ achieving $D_\ell(r) \ge t$, no proper shorter sequence achieves at least t pebbles on r.

Proof of (0). Suppose for contradiction that some move $m_i = (r \to w)$ leaves r. Delete m_i from the sequence. Immediately after index i, relative to D_i, vertex r has discrepancy $+2$, vertex $w \ne r$ has discrepancy -1, and every other vertex has discrepancy 0. We scan indices $s = i+1, \dots, \ell$ in order while maintaining the invariant that after step s, either:

(I) every $v \ne r$ has discrepancy ≥ 0 and r has discrepancy at least $+1$ (zero remaining deficit), or
(II) there is a unique deficit vertex $x_s \ne r$ with discrepancy -1, vertex r has discrepancy $+2$, and every $v \notin \{r, x_s\}$ has discrepancy ≥ 0.

At step $s > i$: if (I) holds after step $s-1$, or (II) holds and m_s does not leave x_{s-1}, then every vertex involved in m_s holds at least as many pebbles as before m_s in σ, so m_s remains legal and preserves (I) or (II) (with $x_s = x_{s-1}$). Suppose (II) holds and $m_s = (x_{s-1} \to y)$ leaves x_{s-1}. If $D_{s-1}(x_{s-1}) \ge 3$, then with discrepancy -1 vertex x_{s-1} still holds at least 2 pebbles, so m_s remains legal and (II) persists with $x_s = x_{s-1}$. If instead $D_{s-1}(x_{s-1}) = 2$, we delete m_s; then x_{s-1} moves from discrepancy -1 to $-1+2 = +1 \ge 0$, while y's discrepancy changes by -1:

- if $y = r$, the $+2$ surplus at r absorbs -1 to leave discrepancy $+1$ at r and discrepancy ≥ 0 on every $v \ne r$, entering (I);
- if $y \ne r$ and y had discrepancy 0 before step s, then y's discrepancy becomes -1, preserving (II) with $x_s = y$ and surplus $+2$ on r;
- if $y \ne r$ and y already had discrepancy $\ge +1$ before step s (from an earlier deleted move out of y), then y's discrepancy becomes ≥ 0, so every $v \ne r$ has discrepancy ≥ 0 and r retains discrepancy $+2 \ge +1$, entering (I).

At the end of the scan, the resulting sequence has length at most $\ell - 1$, remains valid at every step, and ends with at least $D_\ell(r) + 1 \ge t + 1$ pebbles on r, contradicting the minimality of ℓ.

Proof of (1). Suppose some move enters $I \setminus \{r\}$, and let $m_i = (c \to u)$ be the *last* move in σ with $u \in I \setminus \{r\}$. Since I is independent, $c \in C$. After step i, no move enters u.

- If no move after m_i leaves u, deleting m_i leaves u with $D_{i-1}(u) \ge 0$ pebbles for all $s \ge i$ and gives c two extra pebbles from step i onward, yielding a valid sequence of length $\ell - 1$ with at least t pebbles on r, a contradiction.
- Otherwise, let $m_j = (u \to c')$ ($j > i$) be the *first* move after m_i leaving u; necessarily $c' \in C$. For all $i \le s < j$, no move touches u, so $D_s(u) = D_i(u) = D_{i-1}(u) + 1 \ge 1$, and $D_{j-1}(u) \ge 2$ implies $D_{i-1}(u) \ge 1$. If $c' = c$, delete both m_i and m_j: then u holds $D_{i-1}(u) \ge 1$ on $[i, j-1]$ and $D_s(u) + 1$ for $s \ge j$, while c gains $+2$ on $[i, j-1]$ and $+1$ for $s \ge j$, giving a valid sequence of length $\ell - 2$. If $c' \ne c$, then $cc' \in E$ because C is a clique. Replace $m_i = (c \to u)$ by $m_i' = (c \to c')$ and delete m_j. At every step s, c has the exact pebble count $D_s(c)$; u holds $D_{i-1}(u) \ge 1$ on $[i, j-1]$ and $D_s(u) + 1$ for $s \ge j$; and c' gains $+1$ on $[i, j-1]$ and matches $D_s(c')$ for $s \ge j$. All other vertices (including $r \ne u$) are unchanged, giving a valid sequence of length $\ell - 1$, contradiction.

Proof of (2). First assume $r \in C$, and suppose some move has both endpoints in $C \setminus \{r\}$. Let $m_i = (c \to c')$ be the *last* such move. Then $D_i(c') = D_{i-1}(c') + 1 \ge 1$, and if any move leaves c' after m_i, let $m_j = (c' \to z)$ ($j > i$) be the first such move. By (0), $c' \ne r$; by (1), $z \notin I \setminus \{r\}$; and by the choice of m_i as the last move inside $C \setminus \{r\}$, we cannot have $z \in C \setminus \{r\}$. Hence $z = r$, i.e., $m_j = (c' \to r)$. Replacing m_i by $m_i' = (c \to r)$ (valid since $c, r \in C$) and deleting m_j keeps c identical, keeps c' at $D_s(c') - 1 \ge D_i(c') - 1 \ge 0$ on $[i, j-1]$ (as no move leaves c' in (i, j)) and $D_s(c') + 1$ for $s \ge j$, and gives r a $+1$ surplus on $[i, j-1]$ and exact count $D_s(r)$ for $s \ge j$. If no move leaves c' after m_i, deleting m_i suffices. Either case yields a shorter valid sequence, contradiction.

Next assume $r \in I$, and suppose some move $m_i = (c_1 \to c')$ has $c_1 \in C_1$ and $c' \in C$. If no move leaves c' after m_i, deleting m_i contradicts minimality. Otherwise let $m_j = (c' \to z)$ ($j > i$) be the first move after m_i leaving c'. For $i \le s < j$, no move leaves c', so $D_s(c') \ge D_i(c') \ge 1$. By (1), $z \in C \cup \{r\}$. If $z = r$, replace m_i by $(c_1 \to r)$ (valid since $c_1 \in C_1 = N(r)$) and delete m_j. If $z = c_1$, delete both m_i and m_j. If $z \in C \setminus \{c_1\}$, replace m_i by $(c_1 \to z)$ (valid since C is a clique) and delete m_j. In all three cases, pebble counts remain nonnegative everywhere and r ends with at least $D_\ell(r) \ge t$ pebbles in fewer than ℓ moves, contradiction.

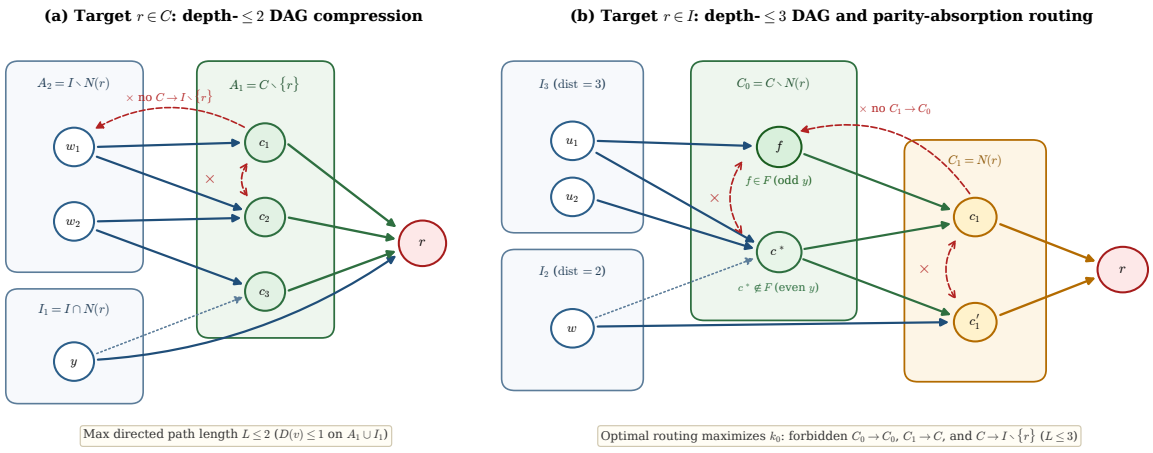


FIGURE 1. Layered DAG compression from the Intra-Clique Move Elimination Lemma (Lemma 2.1). (a) For $r \in C$, optimal sequences compress to depth ≤ 2 ($I \to C \setminus \{r\} \to r$ and $I_1 \to r$). (b) For $r \in I$, optimal sequences compress to depth ≤ 3 ($(I_2 \cup I_3) \to C_0 \to C_1 \to r$ and $I_2 \to C_1$); all intra-clique moves inside C_0 or C_1 and reverse moves $C_1 \to C_0$ or $C \to I \setminus \{r\}$ are eliminated.

Proof of (3). Assume $r \in I$, and suppose some move $m_i = (c_0 \to c_0')$ has $c_0, c_0' \in C_0$. Since $c_0' \notin N(r)$ and no move enters $I \setminus \{r\}$ by (1), the first move $m_j = (c_0' \to z)$ leaving c_0' after m_i (which must exist, else deleting m_i shortens σ) satisfies $z \in C$. Deleting m_i and m_j if $z = c_0$, or replacing m_i by $(c_0 \to z)$ and deleting m_j if $z \neq c_0$, again produces a valid shorter sequence, contradiction.

Finally, checking the surviving edge directions in (0)–(3) confirms that $\vec{H}(G, r)$ is acyclic with maximum directed path length at most 2 for $r \in C$ and at most 3 for $r \in I$. □

Figure 1 illustrates the compressed DAGs $\vec{H}(G, r)$ for both target locations. Lemma 2.1 immediately implies a parity normalization on the independent set $I \setminus \{r\}$.

Lemma 2.2 (Upward Even-to-Odd Boost on $I \setminus \{r\}$)**.** *Let $G = (C \sqcup I, E)$ be a connected split graph with target $r \in V(G)$, and let D be a configuration on $V(G)$. Suppose $u \in I \setminus \{r\}$ satisfies $D(u) = 2m \geq 2$, and define D' by setting $D'(u) := 2m + 1$ and $D'(v) := D(v)$ for all $v \neq u$. Then for every $t \geq 1$, D' is t-solvable for r if and only if D is t-solvable for r.*

Proof. Since $D \leq D'$ pointwise, t-solvability of D implies t-solvability of D'. Conversely, suppose D' is t-solvable for r, and choose a minimum-length valid sequence σ' from D' delivering at least t pebbles to r. By Lemma 2.1(1), no move of σ' enters u. Consequently, σ' executes at most $\lfloor D'(u)/2 \rfloor = \lfloor (2m+1)/2 \rfloor = m = \lfloor D(u)/2 \rfloor$ moves out of u, consuming at most $2m$ pebbles from u. Because $D(v) = D'(v)$ for all $v \neq u$, the exact same sequence σ' is valid from D and delivers the same number of pebbles to r. □

2.2. **Lower-Bound Configurations.** We next record two elementary lower-bound constructions for $\pi(G)$ that interact directly with our conservation bounds in Section 4.

Lemma 2.3 (Unit-Degree Lower Bound)**.** *Let $G = (C \sqcup I, E)$ be a non-complete connected split graph with maximal clique C of size $k \geq 2$. If there exists a vertex $w \in I$ with $\deg(w) = 1$, then $\pi(G) \geq n + 1$.*

Proof. Let $N(w) = \{c\} \subseteq C$. Since $k \geq 2$, there exists $r \in C \setminus \{c\}$. Place 3 pebbles on w, 0 pebbles on c and r, and 1 pebble on each of the $n - 3$ vertices in $V \setminus \{w, c, r\}$, for a total of n pebbles. By Lemma 2.1, any valid sequence reaching r cannot enter $I \setminus \{r\}$ and cannot use edges inside $C \setminus \{r\}$. Every vertex in $I \setminus \{r, w\}$ holds 1 pebble and never moves, w can move at most $\lfloor 3/2 \rfloor = 1$ pebble to c, and every vertex in $C \setminus \{c, r\}$ holds 1 pebble and receives 0 pebbles. Thus c ends with at most 1 pebble, so r cannot be reached. □

Lemma 2.4 (Disjoint-Neighborhood Lower Bound)**.** *Let $G = (C \sqcup I, E)$ be a connected split graph, let $r \in I$ with $C_1 = N(r) \subseteq C$, and suppose $u \in I_3$ (so $N(u) \subseteq C_0 = C \setminus C_1$). Then*

$$\pi(G) \geq n + 6 - |N(u)| - |C_1|.$$

Proof. Set $N_u := N(u) \subseteq C_0$. Since $\{r\}$, $\{u\}$, N_u, and C_1 are pairwise disjoint, their union has size $2 + |N_u| + |C_1|$. Define a configuration D by placing 7 pebbles on u, 0 pebbles on every vertex in $\{r\} \cup N_u \cup C_1$, and 1 pebble on

every remaining vertex in $V \setminus (\{r,u\} \cup N_u \cup C_1)$, so

$$p(D) = 7 + \big(n - 2 - |N_u| - |C_1|\big) = n + 5 - |N_u| - |C_1|.$$

Suppose D could deliver 1 pebble to r, and take a minimum-length sequence σ doing so. By Lemma 2.1, no move enters $I \setminus \{r\}$ and no move goes inside C_0 or C_1. Every vertex in $I \setminus \{r,u\}$ has 1 pebble and cannot move, and every vertex in $C_0 \setminus N_u$ starts with 1 pebble and receives 0 pebbles, so $C_0 \setminus N_u$ sends 0 pebbles to C_1. Vertex u executes at most $\lfloor 7/2 \rfloor = 3$ moves into N_u; because no move occurs within C_0, if u sends a_c pebbles to $c \in N_u$ ($\sum_{c\in N_u} a_c \le 3$), then N_u sends at most $\sum_{c\in N_u} \lfloor a_c/2 \rfloor \le \lfloor 3/2 \rfloor = 1$ pebble into C_1. As C_1 starts with 0 pebbles and receives at most 1 pebble in total, no vertex in C_1 ever holds 2 pebbles, so r cannot be reached. □

3. Sharp t-Pebbling Bounds and the $2t$-Pebbling Frontier

To convert the bounded-depth DAG compression of Lemma 2.1 into an inductive bound on $\pi_t(G)$, we extract a bounded-weight causal sub-configuration for each delivered pebble.

Lemma 3.1 (Causal Sub-Configuration Extraction)**.** *Let $\vec{H} = (V, \vec{E})$ be a directed acyclic graph with target $r \in V$ such that every directed path in $\vec{H}$ ending at r has length at most L. If a pebbling sequence $\sigma = (m_1, \dots, m_\ell)$ on $\vec{H}$ from configuration D places at least 1 pebble on r, then there exists a sub-configuration $D_{\text{used}} \le D$ (pointwise) with $p(D_{\text{used}}) \le 2^L$ that alone moves 1 pebble to r.*

Proof. Treat the initial $p(D)$ pebbles as distinct physical tokens on their starting vertices. Whenever move $m_s = (x \to y)$ executes, it consumes two distinct tokens currently at x and creates one new token at y whose two children in an ancestry tree are the two consumed tokens. At termination, choose one token located at r and let T be its causal binary ancestry tree.

Because $\vec{H}$ is a DAG and time s strictly increases along moves toward the root of T, every token produced by a move m_s can appear at most once in T, and every root-to-leaf path in T traces a directed path in $\vec{H}$ ending at r in reverse. Hence T is a binary tree of depth at most L whose leaves are pairwise distinct initial tokens of D. Since a binary tree of depth at most L has at most 2^L leaves, letting $D_{\text{used}}(v)$ count the leaves of T starting at $v \in V$ gives $D_{\text{used}} \le D$ with $p(D_{\text{used}}) \le 2^L$. Executing the moves corresponding to the internal nodes of T in their exact chronological order from σ consumes only the leaves of T and places the root token on r. □

Proof of Theorem 1.2. Let $d := \operatorname{diam}(G) \in \{0,1,2,3\}$. We first prove the upper bound $\pi_t(G) \le \pi(G) + 2^d(t-1)$ for all $t \ge 1$ by analyzing each diameter:

- *Case $d = 0$.* Here $G \cong K_1$, so $\pi_t(K_1) = t = 1 + 2^0(t-1)$.
- *Case $d = 1$.* Here $G \cong K_n$ ($n \ge 2$), with $\pi(K_n) = n$. We verify $\pi_t(K_n) = n + 2t - 2 = \pi(K_n) + 2^1(t-1)$ directly. For the upper bound, let $p(D) = n + 2t - 2$ and fix $r \in V$. Set $s := D(r)$; if $s \ge t$ we are done, so assume $0 \le s \le t-1$. The $n-1$ vertices in $V \setminus \{r\}$ hold $n + 2t - 2 - s$ pebbles, of which at most $n-1$ are odd remainders, so they deliver $\sum_{v\ne r} \lfloor D(v)/2 \rfloor \ge \lceil (n + 2t - 2 - s - (n-1))/2 \rceil = \lceil t - (s+1)/2 \rceil = t - \lfloor (s+1)/2 \rfloor$ pebbles directly to r. Since $s - \lfloor (s+1)/2 \rfloor = \lfloor s/2 \rfloor \ge 0$, r receives at least $s + t - \lfloor (s+1)/2 \rfloor \ge t$ pebbles.
- *Case $d = 2$.* By Herscovici, Hester, and Hurlbert [6, Theorem 2.7], every connected graph of diameter 2 satisfies $\pi_t(G) \le \pi(G) + 4(t-1) = \pi(G) + 2^2(t-1)$. (Note that direct application of Lemma 3.1 in diameter 2 would yield $8(t-1)$ when $r \in I$ and $C_0 \ne \emptyset$, because $\vec{H}(G,r)$ still has paths of length 3.)
- *Case $d = 3$.* We induct on $t \ge 1$. The base case $t = 1$ holds identically. For $t \ge 2$, let $p(D) = \pi(G) + 8(t-1)$ and fix $r \in V$. Since $p(D) \ge \pi(G)$, there exists a minimum-length sequence σ from D placing at least 1 pebble on r. By Lemma 2.1, σ runs on the DAG $\vec{H}(G,r)$ of maximum directed path length $L \le 3$. By Lemma 3.1, there exists $D_{\text{used}} \le D$ with $p(D_{\text{used}}) \le 2^3 = 8$ that moves 1 pebble to r. The residual configuration $D' := D - D_{\text{used}}$ satisfies $p(D') \ge \pi(G) + 8(t-1) - 8 = \pi(G) + 8(t-2)$, so by the induction hypothesis D' moves $t-1$ pebbles to r. Concatenating the two sequences places t pebbles on r.

Sharpness across all diameters $d \in \{0,1,2,3\}$. For $d = 0$, K_1 attains $t = 1 + 2^0(t-1)$. For $d = 1$ ($n \ge 2$), placing $2t-1$ pebbles on one vertex $u \ne r$, 1 pebble on each of the $n-2$ vertices in $V \setminus \{r,u\}$, and 0 on r gives $n + 2t - 3$ pebbles incapable of delivering t pebbles to r, so $\pi_t(K_n) = n + 2t - 2$.

For $d = 2$, let $m \ge 2$ and let $G = K_m + \{u\}$ consist of a clique $C = \{c_1, \dots, c_m\}$ and one pendant vertex u adjacent only to c_1, so $n = m+1$ and $\operatorname{diam}(G) = 2$. By Lemma 2.3 (for $m \ge 2$, noting $K_2 + \{u\} \cong P_3$), $\pi(G) \ge n + 1 = m + 2$, and since every connected diameter-2 graph has $\pi(G) \le n + 1$ [3], we have $\pi(G) = m + 2$. Placing $4t - 1$ pebbles on u, 1 pebble on each of $c_2, \dots, c_{m-1}$ (empty if $m = 2$), and 0 on c_1, c_m yields $m + 4t - 3 = \pi(G) + 4(t-1) - 1$ pebbles; by Lemma 2.1 for target $r = c_m$, u sends at most $\lfloor (4t-1)/2 \rfloor = 2t - 1$ to c_1, which sends at most $t - 1$ to c_m, while $c_2, \dots, c_{m-1}$ cannot move. Thus $\pi_t(G) = \pi(G) + 4(t-1)$ for all $m \ge 2, t \ge 1$.

For $d = 3$ with $\delta(G) = 1$, let $m \geq 2$ and let $G = K_m + \{r, u\}$ consist of a clique $C = \{c_0, c_1, \ldots, c_{m-1}\}$ together with pendant leaves u adjacent to c_0 and r adjacent to c_1, so $n = m + 2 \geq 4$. By Lemma 2.4 with $|N(u)| = |C_1| = 1$, $\pi(G) \geq n + 6 - 1 - 1 = n + 4$. Conversely, we show every configuration D of $m + 6 = n + 4$ pebbles reaches every target. For any target $c \in C$, if $D(c) = 0$ and $D(c') \leq 1$ for all $c' \in C \setminus \{c\}$, then $D(u) + D(r) \geq (m+6) - (m-1) = 7$, so $\max\{D(u), D(r)\} \geq 4$; hence either u sends ≥ 2 pebbles to c_0 or r sends ≥ 2 pebbles to c_1, delivering at least 1 pebble to c (directly if $c \in \{c_0, c_1\}$, or in one clique step otherwise). By symmetry between r and u, it remains to consider target r, assuming $D(r) = 0$ and $D(c_1) \leq 1$. If $D(c_1) = 1$, u and $C_0 = C \setminus \{c_1\}$ hold $m + 5$ pebbles on m vertices; if $D(u) \leq 1$, $D(C_0) \geq m + 4$ on $m - 1$ vertices so some $c \in C_0$ has ≥ 2 and reaches c_1 (giving 2 at $c_1 \to 1$ at r), while if $D(u) \geq 2$, u sends ≥ 1 to c_0, so C_0 receives $\geq m + 5 - \lceil D(u)/2 \rceil \geq m$ pebbles (when $D(u) \leq 10$, whereas $D(u) \geq 11$ sends ≥ 5 to c_0), reaching c_1. If $D(c_1) = 0$, u and C_0 hold $m + 6$ pebbles; C_0 receives $D(C_0) + \lfloor D(u)/2 \rfloor = m + 6 - \lceil D(u)/2 \rceil$ pebbles, which fails to send 2 pebbles to c_1 only if at most $m + 1$ pebbles reside on C_0, forcing $\lceil D(u)/2 \rceil \geq 5$, i.e., $D(u) \geq 9$; but then u sends ≥ 4 pebbles directly to c_0, which sends ≥ 2 to c_1 and 1 to r. Thus $\pi(K_m + \{r, u\}) = n + 4$. Now for any $t \geq 1$, place $8t - 1$ pebbles on u, 1 pebble on each of $c_2, \ldots, c_{m-1}$ ($n - 4$ vertices), and 0 on c_0, c_1, r, giving $n + 8t - 5$ pebbles. By Lemma 2.1, u sends at most $4t - 1$ to c_0, which sends at most $2t - 1$ to c_1, which sends at most $t - 1$ to r, while $c_2, \ldots, c_{m-1}$ cannot move. Hence $\pi_t(K_m + \{r, u\}) = n + 8t - 4 = \pi(G) + 8(t - 1)$ for every $m \geq 2$ and $t \geq 1$.

For $d = 3$ with $\delta(G) = 3$ (Class 0), consider the family $G_{a,b}$ ($a, b \geq 3$) consisting of a clique $C = C_0 \sqcup C_1$ with $|C_0| = a, |C_1| = b$ and two independent vertices u, r with $N(u) = C_0$ and $N(r) = C_1$, so $n = a + b + 2$ and $\delta(G_{a,b}) = \min\{a, b\} \geq 3$. By Alcón, Gutierrez, and Hurlbert [1] (every connected split graph with $\delta(G) \geq 3$ is Class 0), $G_{a,b}$ has $\pi(G_{a,b}) = a + b + 2$. For completeness, we verify this directly: given $p(D) = a + b + 2$, for any target $c^* \in C_1$ (symmetrically C_0) assume $D(c^*) = 0$ and $D(c) \leq 1$ on $C \setminus \{c^*\}$; if neither C_0 nor $C_1' := C_1 \setminus \{c^*\}$ moves a pebble to c^*, then either $\lfloor D(u)/2 \rfloor = 0$ ($D(u) + D(C_0) \leq a + 1$) or $\lfloor D(u)/2 \rfloor = 1$ with $D(C_0) = 0$ ($D(u) + D(C_0) \leq 3 < a + 1$), and identically $D(r) + D(C_1') \leq \max\{b, 3\} = b$, yielding $p(D) \leq a + b + 1$, contradiction. For target $r \in I$ (symmetrically $u \in I$), assume $D(r) = 0$ and $D(c) \leq 1$ on C_1, set $s_1 := D(C_1) \leq b$, write $D(u) = 2k_u + \epsilon$ ($\epsilon \in \{0, 1\}$), and let $o_0 := o(C_0) \leq a$ and $g(C_0) := \sum_{c \in C_0} \lfloor D(c)/2 \rfloor$. The number of pebbles sent from C_0 to C_1 is $k_0 = g(C_0) + \min(k_u, o_0) + \lfloor \max(0, k_u - o_0)/2 \rfloor$: if $k_u < o_0$, then $2(g(C_0) + k_u) = D(u) + D(C_0) - o_0 - \epsilon \geq b + 1 - s_1$ (giving $k_0 \geq 2$ when $s_1 = 0$ and $k_0 \geq 1$ when $1 \leq s_1 \leq b - 1$; when $s_1 = b$, $D(u) + D(C_0) = a + 2 \geq 5$ forces $k_u \geq 1$ or $g(C_0) \geq 1$, so $k_0 \geq \min(k_u, o_0) + g(C_0) \geq 1$); if $k_u \geq o_0$, then $k_u + \epsilon = \lceil D(u)/2 \rceil \leq \lceil (a + b + 2 - s_1)/2 \rceil$ gives $D(C_0) + k_u = (a + b + 2 - s_1) - (k_u + \epsilon) \geq \lfloor (a + b + 2 - s_1)/2 \rfloor$, which is at least 4 ($k_0 \geq 2$) when $s_1 = 0$ and at least 2 ($k_0 \geq 1$) when $1 \leq s_1 \leq b$, reaching r. In particular, $G_8 := G_{3,3}$ has $\pi(G_8) = 8$. Since $\mathrm{dist}(u, r) = 3$, placing $8t - 1$ pebbles on u fails to deliver t pebbles to r, so $\pi_t(G_8) \geq 8t$; combined with (3), $\pi_t(G_8) = 8t = \pi(G_8) + 8(t - 1)$ for all $t \geq 1$.

The $2t$-pebbling frontier on $G_{4,4}$. Consider $G_{4,4}$ ($n = 10$, $\delta = 4$, diam $= 3$), which is Class 0 so $\pi_1(G_{4,4}) = 10$. Putting $8t - 1$ pebbles on u yields $\pi_t(G_{4,4}) \geq 8t$ for all $t \geq 1$. We prove that $\pi_2(G_{4,4}) = 16 < 18 = \pi_1(G_{4,4}) + 8$. Let $p(D) = 16$.

(1) If the target is $c^* \in C_1$ (and symmetrically $c^* \in C_0$), note first $D(c^*) = 0$ (else $D(c^*) \geq 1$ leaves $15 \geq 10 = \pi_1(G_{4,4})$ pebbles for a second pebble) and $D(c) \leq 1$ for every $c \in C \setminus \{c^*\}$ (if $D(c) \geq 2$, moving $(c \to c^*)$ leaves $14 \geq 10 = \pi_1(G_{4,4})$ pebbles for a second pebble). Let $k_u := \lfloor D(u)/2 \rfloor$ and $k_r := \lfloor D(r)/2 \rfloor$. Then C_0 (4 vertices) sends $s_0 = \min(k_u, D(C_0)) + \lfloor \max(0, k_u - D(C_0))/2 \rfloor$ pebbles to c^*, while $C_1' := C_1 \setminus \{c^*\}$ (3 vertices) sends $s_1' = \min(k_r, D(C_1')) + \lfloor \max(0, k_r - D(C_1'))/2 \rfloor$ pebbles to c^*. Suppose for contradiction that $s_0 + s_1' \leq 1$, so $s_0 \leq 1$ and $s_1' \leq 1$. Observe that $k_u \geq 4$ forces $s_0 \geq 2$ (if $D(C_0) = 0$ then $s_0 = \lfloor k_u/2 \rfloor \geq 2$; if $D(C_0) = 1$ then $s_0 \geq 1 + \lfloor (k_u - 1)/2 \rfloor \geq 2$; if $D(C_0) \geq 2$ then $s_0 \geq 2$). Hence $s_0 \leq 1 \implies k_u \leq 3$:
   - if $k_u \leq 1$, then $D(u) \leq 3$ and $D(C_0) \leq |C_0| = 4$, so $D(u) + D(C_0) \leq 7$;
   - if $k_u = 2$, $s_0 \leq 1$ forces $D(C_0) \leq 1$ (since $D \leq 1$ on C_0, two occupied vertices in C_0 yield 2 pebbles at c^*), so $D(u) \leq 5$ and $D(C_0) \leq 1$, giving $D(u) + D(C_0) \leq 6$;
   - if $k_u = 3$, $s_0 \leq 1$ forces $D(C_0) = 0$, so $D(u) \leq 7$ and $D(u) + D(C_0) \leq 7$.

   Identically on C_1' ($|C_1'| = 3$): $k_r \geq 4$ forces $s_1' \geq 2$, so $s_1' \leq 1 \implies k_r \leq 3$: if $k_r \leq 1$ then $D(r) + D(C_1') \leq 3 + 3 = 6$; if $k_r = 2$ then $D(r) + D(C_1') \leq 5 + 1 = 6$; and if $k_r = 3$ then $D(r) + D(C_1') \leq 7 + 0 = 7$. Consequently, $s_0 + s_1' \leq 1 \implies p(D) = (D(u) + D(C_0)) + (D(r) + D(C_1')) \leq 7 + 7 = 14 < 16$, a contradiction.

(2) If the target is $r \in I$ (and symmetrically $u \in I$), assume $D(r) = 0$ (since $D(r) \geq 1$ leaves $15 \geq 10 = \pi_1(G_{4,4})$ for a second pebble) and $D(c) \leq 1$ on C_1 (since $D(c) \geq 2$ gives 1 on r and leaves $14 \geq 10$ elsewhere). Let $s_1 := D(C_1) \in \{0, 1, 2, 3, 4\}$, write $D(u) = 2k_u + \epsilon$ ($\epsilon \in \{0, 1\}$), and let $o_0 := |\{c \in C_0 : D(c) \text{ odd}\}| \leq 4$. Note that $k_0 = \frac{D(C_0) - o_0}{2} + \min(k_u, o_0) + \lfloor \max(0, k_u - o_0)/2 \rfloor$ pebbles arrive in C_1, which suffices to deliver 2 pebbles to r whenever $k_0 \geq \max(2, 4 - s_1)$. If $k_u < o_0$, then $k_u \leq 3$ and $k_0 = \frac{16 - s_1 - \epsilon - o_0}{2} \geq \lceil (11 - s_1)/2 \rceil \geq \max(2, 4 - s_1)$ for every $s_1 \in \{0, 1, 2, 3, 4\}$. If $k_u \geq o_0$, then $k_0 = \lfloor (D(C_0) + k_u)/2 \rfloor = \lfloor (16 - s_1 - k_u - \epsilon)/2 \rfloor$. Since

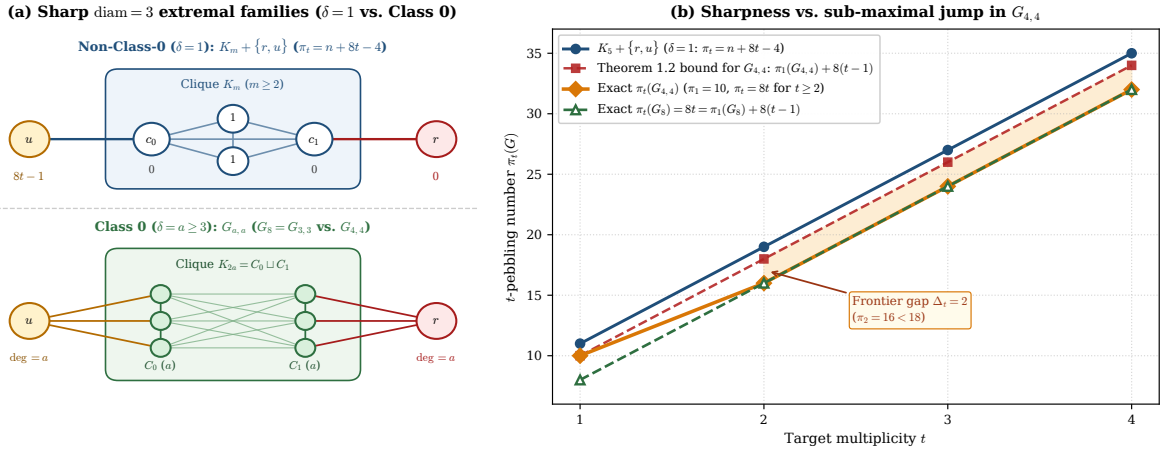


FIGURE 2. Diameter-3 extremal split graphs and the $2t$-pebbling frontier. (a) Non-Class-0 ($\delta = 1$) family $K_m + \{r, u\}$ attaining $\pi_t = n + 8t - 4$, versus Class-0 ($\delta = a \geq 3$) family $G_{a,a}$. (b) Exact t-pebbling numbers for $K_5 + \{r, u\}$ ($n = 7$), $G_8 = G_{3,3}$ ($n = 8$, saturating $8t = \pi_1 + 8(t-1)$), and $G_{4,4}$ ($n = 10$, where $\pi_1 = 10$ jumps sub-maximally to $\pi_t = 8t$ for all $t \geq 2$).

$2k_u + \epsilon + s_1 \leq 16$, we have $k_u + \epsilon \leq 8$ for $s_1 \in \{0, 1\}$ (giving $k_0 \geq 4$ when $s_1 = 0$ and $k_0 \geq 3$ when $s_1 = 1$), $k_u + \epsilon \leq 7$ for $s_1 \in \{2, 3\}$, and $k_u + \epsilon \leq 6$ for $s_1 = 4$ (giving $k_0 \geq 3 \geq 2$ in all cases with $s_1 \geq 2$).

Hence $\pi_2(G_{4,4}) = 16$. Now by induction on $t \geq 2$ using Lemmas 2.1 and 3.1 (extracting D_{used} of size ≤ 8 from $8t$ pebbles leaves at least $8(t-1)$ pebbles, since $8t \geq 10 = \pi_1(G_{4,4})$), $\pi_t(G_{4,4}) \leq 8t$, so $\pi_t(G_{4,4}) = 8t < \pi_1(G_{4,4}) + 8(t-1)$ for all $t \geq 2$ (see Figure 2). □

4. The Full 2-Pebbling Property for Split Graphs

In this section we prove Theorem 1.1. Note first that if $G \cong K_n$ ($n \geq 1$), then $\pi(K_n) = n$ and 2PP holds directly: for $n = 1$, $p(D) + q(D) > 2$ implies $p(D) \geq 2$; for $n \geq 2$, suppose $p(D) + q(D) > 2n$ and $r \in V$. If $D(r) \geq 2$ we are done; if $D(r) = 1$, $q(D) \leq n$ gives $p(D) - 1 \geq 2n - q(D) \geq n = \pi(K_n)$, yielding a second pebble at r; and if $D(r) = 0$, then $q(D) \leq n-1$ forces $p(D) \geq 2n+1-(n-1) = n+2$, so $\sum_{v \neq r} \lfloor D(v)/2 \rfloor \geq (p(D) - q(D))/2 \geq ((n+2)-(n-1))/2 = 3/2$, delivering at least 2 pebbles to r.

Assume henceforth that $G = (C \sqcup I, E)$ is a non-complete connected split graph on $n \geq 3$ vertices with maximal clique C of size $k \geq 2$, and suppose for contradiction that there exist a target $r \in V$ and a configuration D with

$$p(D) + q(D) \geq 2\pi(G) + 1 \tag{4}$$

that is *not* 2-solvable for r.

We first apply standard structural reductions to D:

(1) $D(r) = 0$. Certainly $D(r) \leq 1$. If $D(r) = 1$, since $p(D) \geq q(D)$, (4) gives $2p(D) \geq 2\pi(G) + 1$, so $p(D) \geq \pi(G) + 1$. Removing the single pebble at r leaves $p(D) - 1 \geq \pi(G)$ pebbles, which delivers a second pebble to r, contradiction. Hence $D(r) = 0$.

(2) $D(v) \in \{0, 1\}$ *for all* $v \in N(r)$. Since $D(r) = 0$, every occupied vertex lies in $V \setminus \{r\}$, so $q(D) \leq n-1 \leq \pi(G)-1$. By (4),

$$p(D) \geq 2\pi(G) + 1 - q(D) \geq 2\pi(G) + 1 - (\pi(G) - 1) = \pi(G) + 2.$$

If some $v \in N(r)$ held $D(v) \geq 2$, moving $(v \to r)$ would place 1 pebble at r while leaving $p(D) - 2 \geq \pi(G)$ pebbles on $V \setminus \{r\}$, delivering a second pebble to r, contradiction.

(3) *Odd values on* $I \setminus \{r\}$. Applying Lemma 2.2 repeatedly to every $u \in I \setminus \{r\}$ with $D(u) \geq 2$ even increases $p(D)$ without decreasing $q(D)$ (so (4) persists) while keeping D not 2-solvable for r. Thus we may assume that for every $u \in I \setminus \{r\}$, either $D(u) \in \{0, 1\}$ or $D(u) \geq 3$ is odd.

We split the remainder of the proof into Case A ($r \in C$) and Case B ($r \in I$).

4.1. **Case A:** $r \in C$. Let $A_1 := C \setminus \{r\} \subseteq N(r)$ ($|A_1| = k - 1 \geq 1$), $I_1 := I \cap N(r)$, and $A_2 := I \setminus N(r)$. Because $A_1 \cup I_1 \subseteq N(r)$, every $v \in A_1 \cup I_1$ satisfies $D(v) \in \{0, 1\}$. Partition $A_1 = A_1^1 \sqcup A_1^0$ where $A_1^1 := \{c \in A_1 : D(c) = 1\}$ and $A_1^0 := \{c \in A_1 : D(c) = 0\}$. Note that $N(w) \subseteq A_1$ for every $w \in A_2$.

Claim 4.0.1. For every $w \in A_2$, $D(w) \leq 3$.

Proof. Suppose some $w \in A_2$ has $D(w) \geq 5$; since $D(w)$ is odd, $D(w) = 2j+1$ with $j \geq 2$. Pick $c \in N(w) \subseteq A_1$. If $j \geq 4$ ($D(w) \geq 9$), w sends 4 pebbles to c, which sends 2 pebbles to r, contradiction. Thus $D(w) \in \{5, 7\}$.

Subcase $D(w) = 7$. Then w sends 3 pebbles to any $c \in N(w)$. If any $c \in N(w)$ lies in A_1^1, c reaches 4 and sends 2 to r, contradiction. Hence $N(w) \subseteq A_1^0$. Notice w alone delivers 1 pebble to r via $c \in N(w)$. Set $S_3 := \{u \in A_2 \setminus \{w\} : D(u) = 3\}$. Every $u \in A_2 \setminus \{w\}$ must have $D(u) \leq 3$ (if $D(u) \geq 5$, u sends ≥ 2 to $c' \in N(u)$; if $c' = c$ then c receives $5 \to 2$ at r, and if $c' \neq c$ then c and c' each send 1 to r). Moreover, for every $u \in S_3$, we must have $N(u) \subseteq A_1^0$ (if $c' \in N(u) \cap A_1^1$, then $c' \neq c$ and $u \to c' \to r$ gives a second pebble at r), and the sets $\{N(u) : u \in S_3\} \cup \{N(w)\}$ must be pairwise disjoint: if $u_1, u_2 \in S_3$ shared $c' \in A_1^0$, then sending $u_1, u_2 \to c'$ (or $w, u_1 \to c'$ if $c' \in N(w)$) delivers 2 pebbles to r. Since all $|S_3|+1$ sets are non-empty subsets of A_1^0, we have $|A_1^0| \geq |S_3| + 1$. Counting $p(D) + q(D)$ over $V = \{r, w\} \sqcup S_3 \sqcup (A_2 \setminus (S_3 \cup \{w\})) \sqcup I_1 \sqcup A_1^1 \sqcup A_1^0$ (where w contributes 8, S_3 contributes $4|S_3|$, $(A_2 \setminus (S_3 \cup \{w\})) \cup I_1 \cup A_1^1$ contributes at most $2(n - 2 - |A_1^0| - |S_3|)$, and r, A_1^0 contribute 0) gives

$$p(D) + q(D) \leq 8 + 4|S_3| + 2(n - 2 - |A_1^0| - |S_3|) = 2n + 4 - 2\big(|A_1^0| - |S_3|\big) \leq 2n + 2.$$

Since $p(D) + q(D) \geq 2\pi(G) + 1 \geq 2n + 1$, we must have $|A_1^0| = |S_3| + 1$, which forces all $|S_3| + 1$ disjoint non-empty subsets inside A_1^0 to be singletons; in particular $|N(w)| = 1$ ($\deg(w) = 1$). By Lemma 2.3, $\pi(G) \geq n + 1$, so $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction.

Subcase $D(w) = 5$. Here w sends 2 pebbles to any $c \in N(w)$, delivering 1 pebble to r. By the same reasoning, every $u \in A_2 \setminus \{w\}$ satisfies $D(u) \leq 3$; let $S_3 := \{u \in A_2 \setminus \{w\} : D(u) = 3\}$. No $u \in S_3$ can have a neighbor $c' \in A_1^1$ (since $w \to c$ and $u \to c'$ gives 2 at r whether $c = c'$ or $c \neq c'$), so $N(u) \subseteq A_1^0$ for all $u \in S_3$. Furthermore, no two distinct $u_1, u_2 \in S_3$ share a neighbor $c' \in A_1^0$ (since $u_1, u_2 \to c'$ together with $w \to c$ gives 4 at c if $c' = c$, or 1 each from c and c' if $c' \neq c$). Thus $\{N(u) : u \in S_3\}$ are pairwise disjoint non-empty subsets of A_1^0, so $|A_1^0| \geq |S_3|$. Since w contributes 6 to $p(D) + q(D)$,

$$p(D) + q(D) \leq 6 + 4|S_3| + 2(n - 2 - |A_1^0| - |S_3|) = 2n + 2 - 2\big(|A_1^0| - |S_3|\big).$$

If $|A_1^0| \geq |S_3| + 1$, then $p(D) + q(D) \leq 2n \leq 2\pi(G)$, contradiction. Thus $|A_1^0| = |S_3|$ and $p(D) + q(D) \leq 2n + 2$. If $|S_3| \geq 1$, every $u \in S_3$ has $|N(u)| = 1$, so Lemma 2.3 gives $\pi(G) \geq n + 1$ and $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction. If instead $|S_3| = |A_1^0| = 0$, then $A_1 = A_1^1$; if $|N(w)| \geq 2$, choosing distinct $c_1, c_2 \in N(w) \subseteq A_1^1$ and sending 1 pebble from w to c_1 and 1 to c_2 produces 2 pebbles on r, whereas if $|N(w)| = 1$ ($\deg(w) = 1$), Lemma 2.3 again forces $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction. □

By Claim 4.0.1, every $w \in A_2$ has $D(w) \in \{0, 1, 3\}$. Let $A_2^{(3)} := \{w \in A_2 : D(w) = 3\}$ and set $m := |A_2^{(3)}|$. Counting $p(D) + q(D)$ gives

$$p(D) + q(D) \leq 4m + 2\big(n - 1 - |A_1^0| - m\big) = 2n - 2 + 2\big(m - |A_1^0|\big).$$

Because $p(D) + q(D) \geq 2\pi(G) + 1 \geq 2n + 1$, we must have $2(m - |A_1^0|) \geq 3$, hence

$$m \geq |A_1^0| + 2. \tag{5}$$

Let $T := \{w \in A_2^{(3)} : N(w) \cap A_1^1 \neq \emptyset\}$ and $S := A_2^{(3)} \setminus T$ (so $N(u) \subseteq A_1^0$ for every $u \in S$). Observe that $|T| \leq 2$: if three vertices of T existed, either two have distinct neighbors $c_1 \neq c_2$ in A_1^1 (yielding 2 pebbles at r), or all three share a single neighbor $c^* \in A_1^1$ (placing $1 + 3 = 4$ pebbles on $c^* \to 2$ on r). Moreover, if $|T| = 2$, say $T = \{w_1, w_2\}$, then $N(w_1) \cap A_1^1 = N(w_2) \cap A_1^1 = \{c^*\}$ for a unique $c^* \in A_1^1$.

- *Subcase* $|T| = 2$. Here $w_1, w_2 \to c^*$ already delivers 1 pebble to r, so no two distinct $u_1, u_2 \in S$ can share a neighbor $c \in A_1^0$ (otherwise $u_1, u_2 \to c \to r$ delivers a second pebble to r). Hence $|S| \leq |A_1^0|$, which with (5) forces $|S| = |A_1^0|$, $m = |A_1^0| + 2$, and $p(D) + q(D) \leq 2n + 2$. If $|S| = |A_1^0| \geq 1$, every $u \in S$ has $\deg(u) = 1$; if $|S| = |A_1^0| = 0$, then $A_1^0 = \emptyset$ forces $N(w_1) = N(w_2) = \{c^*\}$, so $\deg(w_1) = 1$. In either event Lemma 2.3 gives $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction.
- *Subcase* $|T| = 1$. Write $T = \{w_1\}$; then w_1 delivers 1 pebble to r via A_1^1, so again no two vertices in S can share a neighbor in A_1^0, yielding $|S| \leq |A_1^0|$ and $m = |S| + 1 \leq |A_1^0| + 1$, contradicting (5).
- *Subcase* $|T| = 0$. Here $S = A_2^{(3)}$ has size $m \geq |A_1^0| + 2$, and $N(u) \subseteq A_1^0$ for all $u \in S$. Choose a map $\phi\colon S \to A_1^0$ with $\phi(u) \in N(u)$ for every $u \in S$ maximizing $\sum_{c \in A_1^0} \lfloor |\phi^{-1}(c)|/2 \rfloor$. Since 2 pebbles cannot reach r, this maximum is at most 1. If the maximum were 0, then $|\phi^{-1}(c)| \leq 1$ for all $c \in A_1^0$, forcing $m \leq |A_1^0|$, contradicting (5). Hence the maximum is 1, so there is a unique vertex $c_0 \in A_1^0$ with $|\phi^{-1}(c_0)| \in \{2, 3\}$ and $|\phi^{-1}(c)| \leq 1$ for all $c \in A_1^0 \setminus \{c_0\}$. If $|\phi^{-1}(c_0)| = 2$, then $m \leq 2 + (|A_1^0| - 1) = |A_1^0| + 1$, contradicting (5). Thus $|\phi^{-1}(c_0)| = 3$, which forces $m = |A_1^0| + 2$, $p(D) + q(D) \leq 2n + 2$, and $|\phi^{-1}(c)| = 1$ for every $c \in A_1^0 \setminus \{c_0\}$. For any $u \in \phi^{-1}(c_0)$, if u had a neighbor $c' \in A_1^0 \setminus \{c_0\}$, re-assigning $\phi(u) := c'$ would yield $|\phi^{-1}(c_0)| = 2$ and $|\phi^{-1}(c')| = 2$, raising the sum to 2 and delivering 2 pebbles to r. Therefore $N(u) = \{c_0\}$ ($\deg(u) = 1$) for each $u \in \phi^{-1}(c_0)$, and Lemma 2.3 gives $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction.

This completes Case A.

4.2. **Case B: $r \in I$.** Let $C_1 = N(r) \subseteq C$ ($|C_1| \geq 1$), $C_0 = C \setminus C_1$, $I_2 = \{w \in I \setminus \{r\} : N(w) \cap C_1 \neq \emptyset\}$, and $I_3 = \{u \in I \setminus \{r\} : N(u) \subseteq C_0\}$. Set $C_1^+ := \{c \in C_1 : D(c) = 1\}$, $C_1^0 := C_1 \setminus C_1^+$, and $z_1 := |C_1^0|$. Also define $C_0^+ := \{c \in C_0 : D(c) \geq 1\}$ and $z_0 := |C_0 \setminus C_0^+|$.

Claim 4.0.2. For every $w \in I_2$, $D(w) \leq 3$.

Proof. Suppose some $w \in I_2$ has $D(w) \geq 5$; since $D(w)$ is odd, $D(w) \in \{5, 7\}$ (if $D(w) \geq 9$, w sends ≥ 4 pebbles to some $c \in N(w) \cap C_1$, delivering 2 to r). Pick $c \in N(w) \cap C_1$. Set $S_{I_2} := \{y \in I_2 \setminus \{w\} : D(y) = 3\}$ and $S_{I_3} := \{u \in I_3 : D(u) = 3\}$.

Subcase $D(w) = 7$. Then w sends 3 pebbles to $c \in N(w) \cap C_1$, delivering 1 pebble to r. Every $c' \in N(w) \cap C_1$ must lie in C_1^0 (otherwise $1 + 3 = 4 \to 2$ at r). Furthermore, every $v \in C_0$ satisfies $D(v) \leq 1$ (else $v \to c$ makes $3 + 1 = 4 \to 2$ at r); every $y \in I_2 \setminus \{w\}$ satisfies $D(y) \leq 3$ (else y sends ≥ 2 to C_1); and every $u \in I_3$ satisfies $D(u) \leq 3$ (else u sends ≥ 2 to $c_0 \in C_0 \to c$). Every $y \in S_{I_2}$ must have $N(y) \cap C_1 \subseteq C_1^0 \setminus \{c\}$ (since hitting C_1^+ or c gives 2 at r), and no two vertices in S_{I_2} can share a neighbor in $C_1^0 \setminus \{c\}$; hence $|S_{I_2}| \leq z_1 - 1$. Likewise, every $u \in S_{I_3}$ must have $N(u) \subseteq C_0^0 := C_0 \setminus C_0^+$, and no two vertices in S_{I_3} share a neighbor in C_0^0 (else 2 pebbles from S_{I_3} enter $c_0 \to c$); hence $|S_{I_3}| \leq z_0$. Counting $p(D) + q(D)$ gives

$$\begin{aligned} p(D) + q(D) &\leq 8 + 4|S_{I_2}| + 4|S_{I_3}| + 2\big(n - 2 - z_1 - z_0 - |S_{I_2}| - |S_{I_3}|\big) \\ &= 2n + 4 - 2(z_1 - |S_{I_2}|) - 2(z_0 - |S_{I_3}|) \leq 2n + 2. \end{aligned}$$

For $p(D) + q(D) \geq 2n + 1$, we must have $z_1 - |S_{I_2}| = 1$ and $z_0 = |S_{I_3}|$. If $|C_1| = 1$, Lemma 2.3 applied to $\deg(r) = 1$ gives $2\pi(G) \geq 2n+2 \geq p(D)+q(D)$. Suppose $|C_1| \geq 2$, and choose $c' \in C_1 \setminus \{c\}$. Because $z_1 - |S_{I_2}| = 1$, either $c' \in C_1^+$ (holding 1 pebble) or $c' \in C_1^0 \setminus \{c\}$ receives 1 pebble from its matched $y \in S_{I_2}$. Similarly, because $z_0 = |S_{I_3}|$, every $c_0 \in C_0$ either holds 1 pebble or receives 1 pebble from its matched $u \in S_{I_3}$. If $N(w) \cap C_0 \neq \emptyset$ (pick $c_0 \in N(w) \cap C_0$), w sends 2 pebbles to c (1 to r) and 1 to c_0 (which sends 1 to c', combining with the 1 on c' to send 1 to r). If $N(w) \cap C_0 = \emptyset$ and $|N(w) \cap C_1| \geq 2$ (pick $c' \in (N(w) \cap C_1) \setminus \{c\}$), w sends 2 to c and 1 to c'. Otherwise $N(w) = \{c\}$, so $\deg(w) = 1$, and Lemma 2.3 gives $2\pi(G) \geq 2n + 2 \geq p(D) + q(D)$, contradiction.

Subcase $D(w) = 5$. Here w sends 2 pebbles to $c \in N(w) \cap C_1$, delivering 1 pebble to r. Again no $y \in I_2 \setminus \{w\}$ can have $D(y) \geq 5$ (otherwise y sends ≥ 2 pebbles to C_1, combining with the 2 pebbles from w to deliver 2 pebbles to r). Let k_0^* be the maximum number of pebbles that $C_0 \cup I_3$ can deliver to C_1 (via $\vec{H}(G, r)$).

- *If $C_1^+ \neq \emptyset$:* Pick $c_1^+ \in C_1^+$. We must have $k_0^* = 0$ (if $k_0^* \geq 1$, sending 1 pebble from C_0 to c_1^+ yields $1 + 1 = 2 \to 1$ at r if $c_1^+ \neq c$, or $1 + 2 + 1 = 4 \to 2$ at r if $c_1^+ = c$). Also every $y \in S_{I_2}$ satisfies $N(y) \cap C_1 \subseteq C_1^0$ (if $c_1^+ \in N(y)$, sending $y \to c_1^+$ and $w \to c$ yields 2 at r) and no two $y_1, y_2 \in S_{I_2}$ share a neighbor in $C_1^0 \setminus \{c\}$, while at most one $y \in S_{I_2}$ hits c (so $|S_{I_2}| \leq z_1$). Since $k_0^* = 0$, every $v \in C_0$ has $D(v) \leq 1$, and every $u \in S_{I_3}$ has $N(u) \subseteq C_0^0$ with pairwise disjoint neighborhoods, so $|S_{I_3}| \leq z_0$. Hence

$$\begin{aligned} p(D) + q(D) &\leq 6 + 4|S_{I_2}| + 4|S_{I_3}| + 2(n - 2 - z_1 - z_0 - |S_{I_2}| - |S_{I_3}|) \\ &= 2n + 2 - 2(z_1 - |S_{I_2}|) - 2(z_0 - |S_{I_3}|) \leq 2n + 2. \end{aligned}$$

  For $p(D) + q(D) \geq 2n + 1$, equality holds: $|S_{I_2}| = z_1$ and $|S_{I_3}| = z_0$. If $|C_1| = 1$, Lemma 2.3 gives $2\pi(G) \geq 2n + 2$. Otherwise pick $c' \in C_1 \setminus \{c\}$: if $c' \in C_1^+$, c' already holds 1 pebble; if $c' \in C_1^0$, c' receives 1 from its matched $y \in S_{I_2}$. Moreover, if $c \in C_1^0$, c also has a matched $y_c \in S_{I_2}$ sending 1 to c. Therefore: if $N(w) \cap C_0 \neq \emptyset$ (pick $c_0 \in N(w) \cap C_0$, which either has 1 pebble or receives 1 from S_{I_3}), sending 1 from w to c (giving $1 + 1 = 2 \to 1$ at r, whether $c \in C_1^+$ or $c \in C_1^0$ via y_c) and 1 from w to c_0 (sending 1 to c', giving $1 + 1 = 2 \to 1$ at r) places 2 pebbles on r; if $N(w) \cap C_0 = \emptyset$ and $|N(w) \cap C_1| \geq 2$, choosing $c' \in (N(w) \cap C_1) \setminus \{c\}$ and sending 1 to c and 1 to c' places 2 pebbles on r; otherwise $N(w) = \{c\}$, so $\deg(w) = 1$, and Lemma 2.3 yields a contradiction.
- *If $C_1^+ = \emptyset$:* Here $z_1 = |C_1| \geq 1$. Note that $k_0^* \leq 1$ (if $k_0^* \geq 2$, C_0 sends 2 pebbles to $c \in C_1$ if w sends 2 to c, giving $4 \to 2$ at r). Let $h_0 := \sum_{v \in C_0} \lfloor D(v)/2 \rfloor$, $e_0 := |\{v \in C_0 : D(v) \geq 2 \text{ even}\}|$, $a := \sum_{u \in I_3} \lfloor D(u)/2 \rfloor$, and fix a routing of a units from I_3 to C_0 realizing k_0^* with odd set $F \subseteq C_0$. Using only the $C_0 \cup I_3$ counting calculation of (7)–(8) and Lemma 4.2(i) proved independently below (without invoking Claim 4.0.2; noting w contributes 6 instead of 2 and S_{I_2} contributes $2|S_{I_2}|$),

$$p(D) + q(D) \leq 2n - 2 + 4 + 2(|S_{I_2}| + k_0^* - z_1) + 2(k_0^* - |C_0 \setminus F|) - (2h_0 - e_0),$$

  and $(2h_0 - e_0) + a = z_0 + 2k_0^* - |C_0 \setminus F|$. If $k_0^* = 1$, we must have $|S_{I_2}| = 0$ (any $y \in S_{I_2}$ puts 1 on $c' \in C_1$; C_0 puts 1 on c', giving 2 at c' and 2 at c if $c' \neq c$, or $2+1+1 = 4$ at c if $c' = c$), so $|S_{I_2}| + k_0^* - z_1 = 1 - z_1$. If $z_1 = 1$, $p(D) + q(D) \leq 2n + 4 - 2|C_0 \setminus F| - (2h_0 - e_0)$; Lemma 2.3 on $\deg(r) = 1$ rules out $\leq 2n + 2$, so $|C_0 \setminus F| = 0$ ($F = C_0$) and $2h_0 - e_0 \leq 1$, forcing $a \geq 1$; any $u \in I_3$ sending to $F = C_0$ has $|N(u)| = 1$ (by Lemma 4.2(i),

which depends only on the maximality of k_0^*), so Lemma 2.4 gives $\pi(G) \geq n+4 \implies 2\pi(G) \geq 2n+8 > 2n+4$. If $z_1 \geq 2$, then $1 - z_1 \leq -1$, giving $p(D) + q(D) \leq 2n + 2 - 2|C_0 \setminus F| - (2h_0 - e_0)$, so for $\geq 2n+1$ we again need $|C_0 \setminus F| = 0$ and $2h_0 - e_0 \leq 1$, forcing $a \geq 1$ and $|N(u)| = 1$; Lemma 2.4 gives $\pi(G) \geq n + 5 - z_1 \geq n+3$ for $z_1 = 2$ (and $p(D) + q(D) \leq 2n$ for $z_1 \geq 3$), contradiction. Finally, if $k_0^* = 0$, then $|S_{I_2}| \leq z_1$ (at most one $y \in S_{I_2}$ hits c, else $w, y_1, y_2 \to c$ gives $4 \to 2$ at r, and no two share a neighbor in $C_1^0 \setminus \{c\}$), $2h_0 - e_0 = 0$, and $a = z_0 - |C_0 \setminus F|$, giving $p(D) + q(D) \leq 2n + 2 - 2(z_1 - |S_{I_2}|) - 2|C_0 \setminus F|$. Thus $p(D) + q(D) \geq 2n+1$ forces $|S_{I_2}| = z_1$ and $|C_0 \setminus F| = 0$ ($F = C_0$). If $z_1 = 1$, Lemma 2.3 on $\deg(r) = 1$ rules out $2n+2$; if $z_1 \geq 2$, every $c' \in C_1$ has a matched $y \in S_{I_2}$ and every $c_0 \in C_0$ holds post-routing count 1, so either $N(w) \cap C_0 \neq \emptyset$ (sending $w \to c$ and $w \to c_0 \to c'$), $|N(w) \cap C_1| \geq 2$ (sending $w \to c$ and $w \to c'$), both giving 2 at r, or $N(w) = \{c\}$ ($\deg(w) = 1$), contradicting Lemma 2.3. □

By Claim 4.0.2 and the odd reduction on $I \setminus \{r\}$, every $w \in I_2$ satisfies $D(w) \in \{0, 1, 3\}$, and every $u \in I_3$ has $D(u) \in \{0, 1\}$ or $D(u) \geq 3$ odd. Let

$$I_2^{(3)} := \{w \in I_2 : D(w) = 3\}, \qquad b := |I_2^{(3)}|, \qquad a := \sum_{u \in I_3} \left\lfloor \frac{D(u)}{2} \right\rfloor.$$

Set $z_{I_2} := |\{w \in I_2 : D(w) = 0\}|$ and $z_{I_3} := |\{u \in I_3 : D(u) = 0\}|$. For $c \in C_0$, define

$$h_0 := \sum_{c \in C_0} \left\lfloor \frac{D(c)}{2} \right\rfloor, \qquad e_0 := \big|\{c \in C_0 : D(c) \geq 2 \text{ is even}\}\big|.$$

Notice the pointwise identity $D(c) - \mathbf{1}_{D(c)\geq 1} = 2\lfloor D(c)/2 \rfloor - \mathbf{1}_{D(c)\geq 2 \text{ even}}$ for every integer $D(c) \geq 0$. Summing over $c \in C_0$ yields

$$\sum_{c \in C_0} D(c) = |C_0^+| + (2h_0 - e_0), \qquad \text{where } 2h_0 - e_0 = \sum_{c \in C_0} \big(D(c) - \mathbf{1}_{D(c)\geq 1}\big) \geq 0. \tag{6}$$

Fix any routing $r_0 \colon E(I_3, C_0) \to \mathbb{Z}_{\geq 0}$ satisfying $\sum_{c \in N(u)} r_0(u, c) = \lfloor D(u)/2 \rfloor$ for all $u \in I_3$ (with empty sums 0 if $C_0 = \emptyset$ or $I_3 = \emptyset$). Define post-routing counts on C_0 by

$$y(c) := D(c) + \sum_{u \in I_3} r_0(u, c) \quad (c \in C_0), \qquad k_0 := \sum_{c \in C_0} \left\lfloor \frac{y(c)}{2} \right\rfloor, \qquad F := \{c \in C_0 : y(c) \text{ is odd}\}.$$

Summing $y(c)$ over $c \in C_0$ gives $2k_0 + |F| = \sum_{c \in C_0} D(c) + a$. Substituting $\sum_{c \in C_0} D(c) = |C_0| - z_0 + 2h_0 - e_0$ from (6) and $|F| = |C_0| - |C_0 \setminus F|$ produces the auxiliary identity

$$(2h_0 - e_0) + a = z_0 + 2k_0 - |C_0 \setminus F|. \tag{7}$$

We now compute $p(D) + q(D) - (2n-2) = \sum_{v \in V \setminus \{r\}} (D(v) + \mathbf{1}_{D(v)\geq 1} - 2)$ by summing across the four blocks C_1, I_2, I_3, C_0:

- On C_1: each $c \in C_1^+$ contributes 0, and each $c \in C_1^0$ contributes -2, totaling $-2z_1$.
- On I_2: each $w \in I_2^{(3)}$ contributes $+2$, each w with $D(w) = 1$ contributes 0, and each w with $D(w) = 0$ contributes -2, totaling $+2b - 2z_{I_2}$.
- On I_3: each $u \in I_3$ with $D(u) = 2k_u + 1 \geq 3$ contributes $2k_u$, each u with $D(u) = 1$ contributes 0, and each u with $D(u) = 0$ contributes -2, totaling $+2a - 2z_{I_3}$.
- On C_0: using (6) and (7),

$$\begin{aligned} \sum_{c \in C_0} \big(D(c) + \mathbf{1}_{D(c)\geq 1} - 2\big) + 2a &= \big(2|C_0^+| + 2h_0 - e_0 - 2|C_0|\big) + 2a \\ &= -2z_0 - (2h_0 - e_0) + 2\big((2h_0 - e_0) + a\big) \\ &= 2k_0 + 2\big(k_0 - |C_0 \setminus F|\big) - (2h_0 - e_0). \end{aligned}$$

Combining all four contributions yields the *Master Conservation Identity*:

$$p(D) + q(D) = 2n - 2 + 2(b + k_0 - z_1) + 2\big(k_0 - |C_0 \setminus F|\big) - (2h_0 - e_0) - 2z_{I_2} - 2z_{I_3}. \tag{8}$$

Notice immediately that $k_0 \leq 3$: since $C_0 \cup C_1$ is a clique, C_0 can deliver k_0 generic pebbles to any chosen vertices of C_1, so $k_0 \geq 4$ places ≥ 4 pebbles on a single $c \in C_1$ and delivers ≥ 2 pebbles to r. Hence $k_0 \in \{0, 1, 2, 3\}$.

Next, choose a routing r_0 maximizing k_0, and independently analyze the $I_2^{(3)} \to C_1$ interface. Let $\psi \colon I_2^{(3)} \to C_1$ be any map with $\psi(w) \in N(w) \cap C_1$ for all $w \in I_2^{(3)}$. Define $y_1(c) := D(c) + |\psi^{-1}(c)|$ for $c \in C_1$, $g_1(\psi) := \sum_{c \in C_1} \lfloor y_1(c)/2 \rfloor$,

and $o_1(\psi) := |\{c \in C_1 : y_1(c) \text{ is odd}\}|$. Because k_0 pebbles from C_0 can be directed first to odd vertices of C_1 and then paired on any single vertex of C_1, the configuration delivers

$$R(\psi) := g_1(\psi) + \min\big(k_0, o_1(\psi)\big) + \left\lfloor \frac{\max\big(0, k_0 - o_1(\psi)\big)}{2} \right\rfloor$$

pebbles to r. Choose ψ to maximize $R(\psi)$, and among such maps to maximize $g_1(\psi)$. Since D is not 2-solvable for r, we have $R(\psi) \le 1$, hence $g_1 := g_1(\psi) \in \{0, 1\}$.

Lemma 4.1 (C_1-Defect Bound)**.** *Write $o_1 := o_1(\psi)$. Then:*

(a) *For $k_0 = 3$, $b = 0$, $z_1 = |C_1|$, and $b + k_0 - z_1 = 3 - |C_1| \le 2$ (with equality only if $|C_1| = 1$).*
(b) *For $k_0 = 2$, $g_1 = 0$, $o_1 \le 1$, and $b + k_0 - z_1 = 2 + o_1 - |C_1| \le 3 - |C_1| \le 2$ (with equality only if $|C_1| = 1$).*
(c) *For $k_0 \in \{0, 1\}$, either $b + k_0 - z_1 \le 1$, or $b + k_0 - z_1 = 2$, which occurs only if either $|C_1| = 1$, or $|C_1| \ge 2$, $k_0 = 0$, $g_1 = 1$, and $o_1 = |C_1|$.*

Proof. Observe that $\sum_{c \in C_1} y_1(c) = 2g_1 + o_1 = |C_1^+| + b = |C_1| - z_1 + b$, so

$$b + k_0 - z_1 = 2g_1 + o_1 + k_0 - |C_1|. \tag{9}$$

If $k_0 = 3$, $R(\psi) \le 1$ forces $g_1 = 0$ and $o_1 = 0$, so $b = |C_1^+| = 0$, $z_1 = |C_1|$, and (9) gives (a). If $k_0 = 2$, $R(\psi) \le 1$ forces $g_1 = 0$ and $o_1 \in \{0, 1\}$, so (9) gives $b + k_0 - z_1 = 2 + o_1 - |C_1| \le 3 - |C_1|$, proving (b). For $k_0 \in \{0, 1\}$: if $g_1 = 0$, then $o_1 \le |C_1| \implies b + k_0 - z_1 = o_1 + k_0 - |C_1| \le k_0 \le 1$. If $g_1 = 1$ and $k_0 = 1$, then $R(\psi) \le 1$ forces $o_1 = 0$, so (9) gives $b + 1 - z_1 = 3 - |C_1| \le 1$ when $|C_1| \ge 2$, and $= 2$ when $|C_1| = 1$. Finally, if $g_1 = 1$ and $k_0 = 0$, (9) gives $b - z_1 = 2 + o_1 - |C_1|$, which is ≤ 1 whenever $o_1 \le |C_1| - 1$ and equals 2 only when $o_1 = |C_1|$. □

We also record the local structure forced by choosing $r_0 \colon E(I_3, C_0) \to \mathbb{Z}_{\ge 0}$ to maximize k_0.

Lemma 4.2 (Optimal $I_3 \to C_0$ Routing Properties)**.** *For any routing r_0 maximizing k_0:*

(i) *If $r_0(u, c) \ge 1$ with $c \in F$, then $N(u) \cap F = \{c\}$.*
(ii) *If $r_0(u, c^*) \ge 2$ with $c^* \in C_0 \setminus F$, then $|N(u) \cap F| \le 1$.*
(iii) *If $r_0(u_1, c^*) \ge 1$ and $r_0(u_2, c^*) \ge 1$ for distinct $u_1, u_2 \in I_3$ and $c^* \in C_0 \setminus F$, then there do not exist distinct $f_1 \in N(u_1) \cap F$ and $f_2 \in N(u_2) \cap F$; consequently, at least one of u_1, u_2 satisfies $|N(u_i) \cap F| \le 1$.*

Proof. In (i), if $c' \in (N(u) \cap F) \setminus \{c\}$, shifting 1 unit of r_0 from (u, c) to (u, c') changes $(y(c), y(c'))$ from (odd, odd) to $(\text{even}, \text{even})$, keeping $\lfloor y(c)/2 \rfloor$ unchanged and increasing $\lfloor y(c')/2 \rfloor$ by 1, contradicting maximality of k_0. In (ii), note $y(c^*) \ge 2$ is even; if distinct $f_1, f_2 \in N(u) \cap F$ existed, shifting 1 unit from (u, c^*) to (u, f_1) and 1 unit to (u, f_2) decreases $\lfloor y(c^*)/2 \rfloor$ by 1 and increases both $\lfloor y(f_1)/2 \rfloor$ and $\lfloor y(f_2)/2 \rfloor$ by 1, raising k_0 by 1. In (iii), if distinct $f_1 \in N(u_1) \cap F$ and $f_2 \in N(u_2) \cap F$ existed, shifting $(u_1, c^*) \to (u_1, f_1)$ and $(u_2, c^*) \to (u_2, f_2)$ decreases $y(c^*)$ by 2 (-1 in k_0) and increases $y(f_1), y(f_2)$ by 1 each ($+2$ in k_0), again raising k_0 by 1. If both $|N(u_1) \cap F| \ge 2$ and $|N(u_2) \cap F| \ge 1$, picking $f_2 \in N(u_2) \cap F$ and $f_1 \in (N(u_1) \cap F) \setminus \{f_2\}$ yields a contradiction; hence $|N(u_1) \cap F| \le 1$ or $N(u_2) \cap F = \emptyset$. □

We now close Case B by checking $a = 0$ and $a \ge 1$ ($k_0 \in \{0, 1, 2, 3\}$).

Closure for $a = 0$ (including $C_0 = \emptyset$). When $a = 0$, $y(c) = D(c)$ for every $c \in C_0$, so $k_0 = h_0$ and $|C_0 \setminus F| = z_0 + e_0$. Thus

$$2\big(k_0 - |C_0 \setminus F|\big) - (2h_0 - e_0) = 2(h_0 - z_0 - e_0) - (2h_0 - e_0) = -e_0 - 2z_0 \le 0,$$

so (8) gives $p(D) + q(D) \le 2n - 2 + 2(b + k_0 - z_1) - e_0 - 2z_0 - 2z_{I_2} - 2z_{I_3}$. If $b + k_0 - z_1 \le 1$, then $p(D) + q(D) \le 2n \le 2\pi(G)$, contradicting (4). If $b + k_0 - z_1 = 2$, then $p(D) + q(D) \le 2n + 2$, and by Lemma 4.1, either $|C_1| = 1$ (so $\deg(r) = 1$ and Lemma 2.3 gives $2\pi(G) \ge 2n + 2 \ge p(D) + q(D)$), or $|C_1| \ge 2$, $k_0 = 0$, $g_1 = 1$, and $o_1 = |C_1|$ (handled uniformly below for all $a \ge 0$).

Closure of the exceptional boundary $b + k_0 - z_1 = 2$ with $|C_1| \ge 2$, $k_0 = 0$, $g_1 = 1$, $o_1 = |C_1|$. Note this closure applies for both $a = 0$ and $a \ge 1$. Since $k_0 = 0$, we have $2h_0 - e_0 = 0$ and (8) gives $p(D) + q(D) \le 2n + 2 - 2|C_0 \setminus F|$. For $p(D) + q(D) \ge 2n + 1$, we must have $|C_0 \setminus F| = 0$ ($F = C_0$), meaning every $c_0 \in C_0$ holds $y(c_0) = 1$. Since $g_1 = 1$ and $o_1 = |C_1|$, there is a unique $c^* \in C_1$ with $y_1(c^*) = 3$ (so $|\psi^{-1}(c^*)| \in \{2, 3\}$), and every $c' \in C_1 \setminus \{c^*\}$ has $y_1(c') = 1$. By maximality of $R(\psi)$, no $w \in \psi^{-1}(c^*)$ has any neighbor in $C_1 \setminus \{c^*\}$. If some $w \in \psi^{-1}(c^*)$ had a neighbor $c_0 \in C_0$, picking $c' \in C_1 \setminus \{c^*\}$ and redirecting $w \to c_0$ (using $y(c_0) = 1$ to send 1 pebble from c_0 to c', combining with $y_1(c') = 1$ to send 1 to r) while keeping the remaining $|\psi^{-1}(c^*)| - 1$ units on c^* (leaving $y_1(c^*) - 1 = 2 \to 1$ at r) delivers 2 pebbles to r, contradiction. Hence $N(w) = \{c^*\}$ ($\deg(w) = 1$) for every $w \in \psi^{-1}(c^*)$, and Lemma 2.3 gives $2\pi(G) \ge 2n + 2 \ge p(D) + q(D)$, contradiction.

Closure for $a \ge 1$ ($k_0 \in \{0, 1, 2, 3\}$). Note $a \ge 1$ implies $I_3 \ne \emptyset$ and $C_0 \ne \emptyset$. We examine each k_0:

- $k_0 = 0$. Here $2h_0 - e_0 = 0$. By Lemma 4.1(c), either $b + k_0 - z_1 \le 1$ (giving $p(D) + q(D) \le 2n - 2|C_0 \setminus F| \le 2n \le 2\pi(G)$), or $|C_1| \ge 2, g_1 = 1, o_1 = |C_1|$ (closed above), or $|C_1| = 1$ and $b + k_0 - z_1 = 2$. In the last case, (8) gives $p(D) + q(D) \le 2n + 2 - 2|C_0 \setminus F|$; since Lemma 2.3 ($\deg(r) = 1$) gives $2\pi(G) \ge 2n + 2 \ge p(D) + q(D)$, this already contradicts (4).
- $k_0 = 1$. By Lemma 4.1(c):
  - If $|C_1| \ge 2$, then $b + 1 - z_1 \le 1$, so (8) gives $p(D) + q(D) \le 2n + 2 - 2|C_0 \setminus F| - (2h_0 - e_0)$. For $p(D) + q(D) \ge 2n + 1$, we must have $|C_0 \setminus F| = 0$ ($F = C_0$). Since $a \ge 1$, some $u \in I_3$ sends at least 1 unit to $F = C_0$; by Lemma 4.2(i), $|N(u)| = 1$, so Lemma 2.3 yields $2\pi(G) \ge 2n + 2 \ge p(D) + q(D)$, contradiction.
  - If $|C_1| = 1$, then $b + 1 - z_1 \le 2$, so (8) gives $p(D) + q(D) \le 2n + 4 - 2|C_0 \setminus F| - (2h_0 - e_0)$. Since $\deg(r) = 1$, Lemma 2.3 gives $2\pi(G) \ge 2n + 2$, forcing $|C_0 \setminus F| = 0$ ($F = C_0$) for $p(D) + q(D) \ge 2n + 3$. Then any $u \in I_3$ sending ≥ 1 unit to $F = C_0$ has $|N(u)| = 1$ by Lemma 4.2(i), and Lemma 2.4 gives $\pi(G) \ge n + 6 - 1 - 1 = n + 4$, so $2\pi(G) \ge 2n + 8 > 2n + 4 \ge p(D) + q(D)$, contradiction.
- $k_0 = 2$. By Lemma 4.1(b), $b + 2 - z_1 \le 3 - |C_1|$, so (8) gives

$$p(D) + q(D) \le 2n + 8 - 2|C_1| - 2|C_0 \setminus F| - (2h_0 - e_0).$$

  If $|C_0 \setminus F| \ge 2$, then $p(D) + q(D) \le 2n + 4 - 2|C_1|$, which is $\le 2n \le 2\pi(G)$ when $|C_1| \ge 2$ and $\le 2n + 2 \le 2\pi(G)$ when $|C_1| = 1$ (by Lemma 2.3). If $|C_0 \setminus F| = 0$ ($F = C_0$), Lemma 4.2(i) gives $u \in I_3$ with $|N(u)| = 1$, so Lemma 2.4 gives $2\pi(G) \ge 2n + 10 - 2|C_1| > 2n + 8 - 2|C_1| \ge p(D) + q(D)$. If $|C_0 \setminus F| = 1$ (write $C_0 \setminus F = \{c^*\}$), then $p(D) + q(D) \le 2n + 6 - 2|C_1| - (2h_0 - e_0)$. Whenever $2h_0 - e_0 \ge 2$, this is at most $2n + 4 - 2|C_1|$ (contradiction as above). Otherwise $2h_0 - e_0 \le 1$, so (7) gives $a = z_0 + 4 - 1 - (2h_0 - e_0) \ge 2$. Either some $u \in I_3$ sends at least 1 unit to F (giving $|N(u) \cap F| = 1$ by Lemma 4.2(i)), or all $a \ge 2$ units land on c^* (so either one u sends ≥ 2 units to c^* or two distinct u_1, u_2 each send ≥ 1 unit to c^*; by Lemma 4.2(ii)–(iii), at least one such u satisfies $|N(u) \cap F| \le 1$). In all cases there exists $u \in I_3$ with $|N(u)| \le 1 + |C_0 \setminus F| = 2$, and Lemma 2.4 gives $2\pi(G) \ge 2n + 8 - 2|C_1| > 2n + 6 - 2|C_1| \ge p(D) + q(D)$, contradiction.
- $k_0 = 3$. By Lemma 4.1(a), $b + 3 - z_1 = 3 - |C_1|$, so (8) gives

$$p(D) + q(D) \le 2n + 10 - 2|C_1| - 2|C_0 \setminus F| - (2h_0 - e_0).$$

  If $|C_0 \setminus F| \ge 3$, then $p(D) + q(D) \le 2n + 4 - 2|C_1| \le 2\pi(G)$ (using Lemma 2.3 if $|C_1| = 1$). If $|C_0 \setminus F| = 0$ ($F = C_0$), Lemma 4.2(i) gives $u \in I_3$ with $|N(u)| = 1$, and Lemma 2.4 gives $2\pi(G) \ge 2n + 10 - 2|C_1| \ge p(D) + q(D)$, contradicting (4). If $|C_0 \setminus F| = 1$, then $p(D) + q(D) \le 2n + 8 - 2|C_1| - (2h_0 - e_0)$; if $2h_0 - e_0 \ge 4$, this is $\le 2n + 4 - 2|C_1| \le 2\pi(G)$, while if $2h_0 - e_0 \le 3$, (7) gives $a = z_0 + 6 - 1 - (2h_0 - e_0) \ge 2$, and the identical Lemma 4.2 check produces $u \in I_3$ with $|N(u)| \le 2$, so Lemma 2.4 yields $2\pi(G) \ge 2n + 8 - 2|C_1| \ge p(D) + q(D)$, contradiction. Finally, if $|C_0 \setminus F| = 2$ (write $C_0 \setminus F = \{c_1^*, c_2^*\}$), then $p(D) + q(D) \le 2n + 6 - 2|C_1| - (2h_0 - e_0)$. If $2h_0 - e_0 \ge 2$, this is $\le 2n + 4 - 2|C_1| \le 2\pi(G)$. If $2h_0 - e_0 \le 1$, (7) gives $a = z_0 + 6 - 2 - (2h_0 - e_0) \ge 3$. Either some $u \in I_3$ sends at least 1 unit to F (giving $|N(u) \cap F| = 1$ by Lemma 4.2(i)), or all $a \ge 3$ units land on $\{c_1^*, c_2^*\}$. By the pigeonhole principle, at least one $c^* \in \{c_1^*, c_2^*\}$ receives at least 2 units: either a single $u \in I_3$ sends ≥ 2 units to c^*, or two distinct $u_1, u_2 \in I_3$ each send ≥ 1 unit to c^*. By Lemma 4.2(ii)–(iii), there exists $u \in I_3$ with $|N(u) \cap F| \le 1$, hence $|N(u)| \le 1 + |C_0 \setminus F| = 3$. Lemma 2.4 gives $\pi(G) \ge n + 6 - 3 - |C_1| = n + 3 - |C_1|$, so $2\pi(G) \ge 2n + 6 - 2|C_1| \ge p(D) + q(D)$, contradicting (4).

This exhausts all possibilities and completes the proof of Theorem 1.1.

5. Graham's Pebbling Conjecture for Split Graphs

By Theorem 1.1, every connected split graph G satisfies both 2PP and O2PP. Combining Theorem 1.1 with the classical 2PP product theorems already resolves four of the five target families in Theorem 1.3:

- If $H \cong K_m$ is a complete graph, $\pi(G \,\square\, K_m) \le m\pi(G) = \pi(G)\pi(K_m)$ by [2, Theorem 5(i)].
- If $H = T$ is a tree, $\pi(G \,\square\, T) \le \pi(G)\pi(T)$ by Moews [10] and Snevily–Foster [11].
- If H is a complete bipartite graph, $\pi(G \,\square\, H) \le \pi(G)\pi(H)$ by Feng–Kim [4].
- If H is a complete multipartite graph, $\pi(G \,\square\, H) \le \pi(G)\pi(H)$ by Wang [12].

For hypercubes $H = Q_d$ ($d \ge 1$), caution is required when invoking [2, Theorem 5(ii)], which states that $G \,\square\, K_2$ inherits 2PP *provided* $\pi(G \,\square\, K_2) = 2\pi(G)$. As verified in Section 6 (and shown in Figure 3(b)), connected split graphs can exhibit a strict product drop $\pi(G \,\square\, K_2) < 2\pi(G)$, such as $G = K_3 + \text{leaf}$ ($\pi(G) = 5$ yet $\pi(G \,\square\, K_2) = 9 < 10$). To establish $\pi(G \,\square\, Q_d) \le 2^d \pi(G)$ without assuming equality at intermediate steps, we lift O2PP relative to an explicit budget M.

Lemma 5.1 (Cube Lifting of O2PP)**.** *Let G be a connected graph and let M be an integer satisfying $M \ge \max\{\pi(G), |V(G)|\}$ such that (G, M) has the M-*odd 2-pebbling property*: every configuration D on G with*

TABLE 1. Exhaustive census of all 164 connected split graphs on $n \leq 7$ vertices by diameter d, showing 2PP compliance, Class 0 counts, and exact attainment of the Theorem 1.2 bound at $t \in \{2, 3\}$.

Diameter d	Total connected splits	Satisfying 2PP	Class 0 ($\pi(G) = n$)	Sharp Eq. (3) ($t = 2, 3$)
0	1	1	1	1
1	6	6	6	6
2	92	92	52	74
3	65	65	0	65
Total	164	164	59	146

$p(D) + o(D) > 2M$ is 2-solvable for every target in $V(G)$. Then $(G \,\square\, K_2, 2M)$ has the $(2M)$-odd 2-pebbling property and $\pi(G \,\square\, K_2) \leq 2M$.

Proof. Write $V(G \,\square\, K_2) = G_0 \sqcup G_1$ for the two vertex-disjoint copies of G joined by matching edges $(v, 1)(v, 0)$, and fix any target $r_0 = (r, 0) \in G_0$ with twin $r_1 = (r, 1) \in G_1$. For any configuration $D = (D_0, D_1)$, write $p_i := p(D_i)$ and $o_i := o(D_i) \leq |V(G)| \leq M$.

First suppose $p_0 + p_1 = 2M$. If $p_0 \geq M \geq \pi(G)$, G_0 alone delivers 1 pebble to r_0. If $p_0 < M$, consider the maximum number $(p_1 - o_1)/2$ of pebbles transferable from G_1 to G_0 across matching edges. If $(p_1 - o_1)/2 \geq M - p_0$, moving $M - p_0$ pebbles from G_1 to G_0 yields M pebbles on G_0, hence 1 pebble on r_0. Otherwise $(p_1 - o_1)/2 \leq M - p_0 - 1$, i.e., $p_1 - o_1 \leq 2M - 2p_0 - 2$; subtracting from $p_1 = 2M - p_0$ gives $o_1 \geq p_0 + 2$, so $p_1 + o_1 = 2M - p_0 + o_1 \geq 2M + 2 > 2M$. By hypothesis on (G_1, M), G_1 moves 2 pebbles to r_1, whence 1 move puts 1 pebble on r_0. Thus $\pi(G \,\square\, K_2) \leq 2M$.

Next suppose $p_0 + o_0 + p_1 + o_1 \geq 4M + 1$.

- If $p_0 + o_0 > 2M$, then (G_0, M) moves 2 pebbles to r_0.
- If $p_0 \geq M$ and $p_0 + o_0 \leq 2M$, then $p_1 + o_1 \geq 2M + 1 > 2M$. Thus G_0 moves 1 pebble to r_0 and G_1 moves 2 pebbles to r_1 (hence 1 to r_0), giving 2 pebbles on r_0.
- If $p_0 < M$, then since $o_0 \leq p_0$ and $o_1 \leq |V(G)| \leq M$,

$$p_1 - o_1 = (p_0 + o_0 + p_1 + o_1) - p_0 - o_0 - 2o_1 \geq 4M + 1 - 2p_0 - 2M = 2M + 1 - 2p_0.$$

Because $p_1 - o_1$ is even, $(p_1 - o_1)/2 \geq M - p_0 + 1 > M - p_0$. Move $M - p_0$ pebbles from G_1 to G_0 across matching edges (consuming $2(M - p_0)$ pebbles in G_1). Then G_0 receives $p_0' = p_0 + (M - p_0) = M \geq \pi(G)$ pebbles and delivers 1 pebble to r_0. Crucially, removing an even number of pebbles from any vertex of G_1 leaves the parity of every vertex in G_1 unchanged, so the residual configuration D_1' on G_1 satisfies $o(D_1') = o_1$ and

$$p(D_1') + o(D_1') = p_1 - 2(M - p_0) + o_1 \geq (4M + 1 - p_0 - o_0) - 2M + 2p_0 = 2M + 1 + (p_0 - o_0) > 2M.$$

By hypothesis on (G_1, M), G_1 moves 2 pebbles to r_1 and hence 1 additional pebble to r_0, placing 2 pebbles on r_0 in total. □

Proof of Theorem 1.3. For trees, complete graphs, complete bipartite graphs, and complete multipartite graphs, the bound $\pi(G \,\square\, H) \leq \pi(G)\pi(H)$ follows directly by combining Theorem 1.1 with [2, 10, 11, 4, 12]. For $H = Q_d$ ($d \geq 0$), let $M_0 := \pi(G) \geq |V(G)|$. By Theorem 1.1, (G, M_0) satisfies 02PP. Writing $Q_d = Q_{d-1} \,\square\, K_2$ and setting $M_d := 2^d \pi(G)$, we have $M_d \geq 2^d |V(G)| = |V(G \,\square\, Q_d)|$ at every step, so induction on d via Lemma 5.1 gives $\pi(G \,\square\, Q_d) \leq 2^d \pi(G) = \pi(G)\pi(Q_d)$ (since $\pi(Q_d) = 2^d$ [2]). □

6. Exhaustive Computational Checks and Open Directions

Although Theorems 1.1–1.3 are proved purely mathematically without computer assistance, we implemented an independent deterministic verification suite (`code/verify_split_pebbling.py`) to stress-test Lemma 2.1, Theorems 1.1–1.2, and our extremal constructions across all small connected split graphs.

(1) *Cross-check of Lemma 2.1.* Across all 34 non-complete connected split graphs on $n = 6$ vertices and all 6 target choices (204 graph-target pairs, 3 adversarial/random configurations each, 612 instances total), the maximum number of deliverable pebbles computed by an unrestricted bidirectional-edge MILP (enforcing acyclicity via topological potentials) matched both the restricted depth-≤ 3 DAG MILP and an exact parity-bitmask dynamic program in 100% of cases.

(2) *Exhaustive census on $n \leq 7$.* Enumerating all 164 connected split graphs on $1 \leq n \leq 7$ vertices (1, 1, 2, 5, 12, 35, 108 graphs for $n = 1, \ldots, 7$), we computed exact $\pi_1(G)$ and checked 2PP ($p(D) + q(D) = 2\pi(G) + 1$) and Theorem 1.2 for $t \in \{2, 3\}$ exhaustively across all automorphism orbits. Table 1 summarizes the diameter distribution. Notably, *zero* of the 65 diameter-3 split graphs on $n \leq 7$ are Class 0 (making G_8 at $n = 8$ the

smallest Class-0 diameter-3 split graph), and *all* 65 diameter-3 split graphs on $n \le 7$ attain exact equality $\pi_t(G) = \pi(G) + 8(t-1)$ for $t \in \{2,3\}$.

(3) *Extremal families and product drop.* Exact symmetry-reduced verification confirms $\pi_t(K_m + \{u\}) = m + 4t - 2$ $(2 \le m \le 5)$, $\pi_t(K_m + \{r,u\}) = n + 8t - 4$ $(2 \le m \le 5)$, $\pi_t(G_8) = 8t$ $(t \in \{1,2,3\})$, and $\pi_1(G_{4,4}) = 10$ with $\pi_2(G_{4,4}) = 16$ and $\pi_3(G_{4,4}) = 24$. Moreover, labeling the paw graph $K_3 + \text{leaf}$ with clique $\{0,2,3\}$, degree-3 hub 0, and pendant leaf 1 adjacent only to 0, we verify $\pi((K_3 + \text{leaf}) \,\square\, K_2) = 9 < 10$: the 8-pebble witness $D((1,1)) = 7$, $D((2,1)) = 1$ fails target $r = (3,0)$ (whereas every 9-pebble configuration succeeds across all targets).

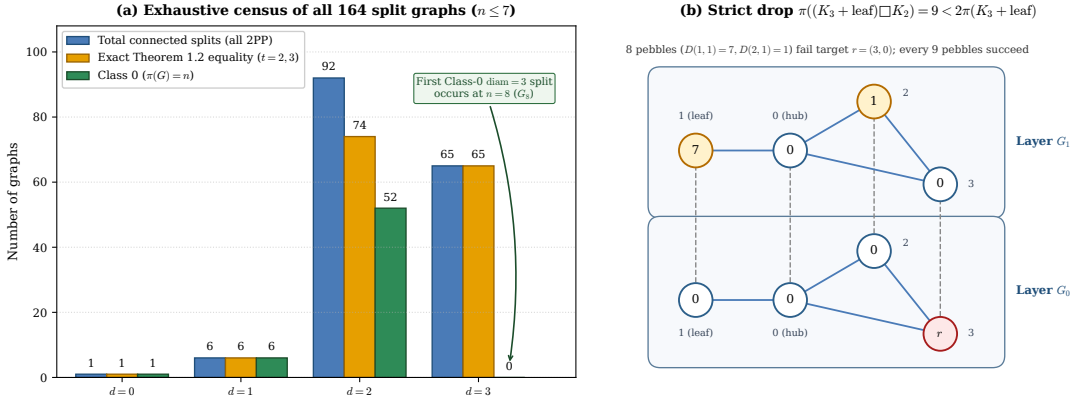


FIGURE 3. (a) Diameter breakdown of all 164 connected split graphs on $n \le 7$ vertices (Table 1): every graph satisfies 2PP, all 65 diameter-3 graphs saturate Theorem 1.2, and none is Class 0 until $n = 8$ (G_8). (b) Strict product drop on $(K_3 + \text{leaf}) \,\square\, K_2$ (clique $\{0,2,3\}$ with hub 0 and leaf 1 adjacent to 0): the 8-pebble configuration $D((1,1)) = 7$, $D((2,1)) = 1$ fails target $r = (3,0)$, whereas every 9-pebble configuration succeeds, giving $\pi((K_3 + \text{leaf}) \,\square\, K_2) = 9 < 2\pi(K_3 + \text{leaf}) = 10$.

Three natural open directions remain. First, split graphs are a proper subclass of chordal graphs, and it remains open whether all connected chordal graphs satisfy 2PP (or whether a chordal Lemke graph exists). Second, characterizing precisely which Class-0 diameter-3 split graphs satisfy $\pi_t(G) = 8t$ for all $t \ge 2$ versus $\pi_t(G) = n + 8(t-1)$ (or intermediate regimes like $G_{4,4}$) would illuminate the full t-pebbling spectrum of dense diameter-3 graphs. Third, verifying Graham's conjecture $\pi(G \,\square\, H) \le \pi(G)\pi(H)$ when *both* G and H are arbitrary connected split graphs remains an inviting next step now that every split graph is known to satisfy 2PP.

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