Arithmetic-Step and Exact-Rational Finite-Witness Certification for Hitting Time Index Bounds, the Degree Threshold, and Lollipop Counterexamples

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https://github.com/lluiseriksson/hitting-time-index-certificates

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Abstract

We present a machine-checked arithmetic kernel and zero-float exact rational finite-witness certificate suite (HT-INDEX-CERT-1.0) auditing key algebraic steps and concrete graph witnesses from Palacios, Tamura, Wiberg, and Petojević (arXiv:2611.34295v1, Nov. 2026) on the hitting time index $\mathrm{HT}(G) = \sum_{1\le i<j\le n} \max(E_iT_j, E_jT_i)$ of simple connected graphs. Our artifact combines an 18-theorem constructive Lean 4.35.0 arithmetic kernel (0 sorry, 0 custom axioms, 0 Classical.choice, 0 warnings), an AST-audited zero-float Python verifier over $\mathbb{Z}$ and $\mathbb{Q}$ solving exact rational hitting-time linear systems, 16 canonical JSON certificates (C01–C16) covering 15 finite benchmark graphs plus one asymptotic closed-form point, and 39 unit and forgery-rejection tests. Strictly distinguishing Lean's decoupled natural-number arithmetic/monotonicity lemmas from finite-instance rational linear-system audits, we confirm the K_6 minimality equality, the K_4-plus-two-pendants non-comparability witness ($262/3 > 82$), finite threshold witnesses separating $K_{5,5}$ from $2K_r + e$ ($r \in \{5,7\}$), the minimal disproof $\mathrm{HT}(P_6) = 223 < 224 = \mathrm{HT}(L(6,4))$, the exact rational lollipop values of Table 1 of [1] through $n = 10$, and the $L(4k, 2k)$ quintic lower-bound formulas and finite solves.

1 Introduction and Audited Scope

For a finite simple connected graph $G = (V, E)$ with $n = |V| \ge 2$ and $m = |E|$, let E_iT_j denote the expected hitting time from vertex i to vertex j of the simple random walk on G. The *hitting time index* [1] is

$$\mathrm{HT}(G) = \sum_{1\le i<j\le n} \max(E_iT_j, E_jT_i). \tag{1}$$

By Chandra's commute-time identity $E_iT_j + E_jT_i = 2mR_{ij}$ (where R_{ij} is the effective resistance between i and j, and $R(G) = \sum_{i<j} R_{ij}$ is the Kirchhoff index), $\mathrm{HT}(G)$ satisfies the universal sandwich $mR(G) \le \mathrm{HT}(G) \le 2mR(G)$.

Palacios et al. [1] establish eight propositions (Propositions 1–8) sharpening tree, regular, diameter, and degree-ordered bounds, prove the conjecture that K_n uniquely minimizes $\mathrm{HT}(G)$ among connected n-vertex graphs, establish a sharp minimum-degree threshold at $\delta = \lfloor n/2 \rfloor$ separating $O(n^3)$ from $\Omega(n^4)$ growth, and disprove the conjecture that the path P_n maximizes $\mathrm{HT}(G)$ by exhibiting lollipop graphs $L(n, k)$ (a k-clique K_k joined to a pendant path of $n - k$ vertices) with $\mathrm{HT}(L(6, 4)) = 224 > 223 = \mathrm{HT}(P_6)$ and proving $\mathrm{HT}(L(4k, 2k)) = \Omega(n^5)$.

This note delivers an independent, fully reproducible machine-audited verification package (HT-INDEX-CERT-1.0) with strict separation of scope:

1. **Constructive Lean 4.35.0 kernel** (HittingTimeIndexCert.lean): 18 universally quantified natural-number algebraic/combinatorial theorems and exact integer witness theorems (0 sorry, 0 Classical.choice, audited via #print axioms).

2. **AST-audited zero-float exact rational verifier** (ht_index_verifier.py): solves the $(n-1)\times(n-1)$ hitting-time linear systems $(I - P_{-j})h^{(j)} = \mathbf{1}$ over $\mathbb{Q}$ using Gaussian elimination with fractions.Fraction (zero floating-point operations) across four verification modes.

2 Constructive Lean 4.35.0 Kernel

All 18 theorems in lean/HittingTimeIndexCert.lean compile under Lean 4.35.0 with zero warnings and depend exclusively on at most propext and Quot.sound (zero Classical.choice, zero sorry). Table 1 records the exact quantified Lean statements and their algebraic role in [1]. We emphasize the exact scope boundary: Lean formalizes decoupled $\mathbb{N}$-arithmetic identities, monotonicity steps, and exact integer comparisons (for example, Theorems 3, 10, and 11 take $2m \le n(n-1)$, $R \le 3n-1$, and $n^2 \le 8R$ as explicit numerical hypotheses without defining graphs, Markov chains, or effective resistance inside Lean), while the finite-graph hitting-time linear systems and resistance evaluations are independently audited over $\mathbb{Q}$ on

15 finite witness graphs by the zero-float Python verifier in Section 3.

3 Zero-Float Exact Rational Verification Modes

The Python verifier `src/ht_index_verifier.py` is statically AST-audited by `pytest` to guarantee zero `float` literals, zero calls to `float`, and zero imports of `math`, `numpy`, `scipy`, or `decimal`. Every rational field is encoded as a lowest-terms pair $[p, q]$ with $q > 0$ and $\gcd(p, q) = 1$, and all boolean/integer fields reject Python `bool`-as-`int` coercion.

Mode 1 (`exact_graph_ht_and_bounds`, `C01`–`C04`). For any connected simple graph G, the verifier solves the exact hitting-time matrix $E_i T_j \in \mathbb{Q}$ over all target vertices j, derives exact $R_{ij} = (E_i T_j + E_j T_i)/(2m)$ and $R(G) = \sum_{i<j} R_{ij}$, and checks:

- Eq. (1) sandwich $mR(G) \le \mathrm{HT}(G) \le 2mR(G)$;
- Prop. 1 for trees: $\mathrm{HT}(T) \ge W_2(T) \ge (H(T) + W_2(T))/2$, where d_{ij} is shortest-path graph distance, $W_2(T) = \sum_{i<j} d_{ij}^2$, and $H(T) = \sum_{i<j} d_{ij}^{-1}$;
- Prop. 2 bounds $m(n-1) \le \mathrm{HT}(G) \le mn(n^2-1)/3$;
- Prop. 3 (K_n minimality): $\mathrm{HT}(G) \ge \mathrm{HT}(K_n) + \binom{n}{2} - m \ge n(n-1)^2/2$, with equality on K_6 (`C01`: $\mathrm{HT}(K_6) = 75$);
- Prop. 4 degree-ordered lower bound: for sorted degrees $d_1 \le \cdots \le d_n$, $\mathrm{HT}(G) \ge m \sum_{k=1}^{n-1} (n-k)/d_k$;
- Prop. 6 regular-graph bound: whenever G is regular ($d_1 = d_n$, e.g. C_8 in `C03`), $\mathrm{HT}(G) \le \binom{n}{2} \cdot 3n^2$;
- Prop. 7 diameter bound $\mathrm{HT}(G) \le 2m\binom{n}{2}\mathrm{diam}(G)$.

Certificate `C04` certifies the exact non-comparability witness K_4 with two pendant vertices attached to distinct clique vertices ($n = 6, m = 8$, degree sequence $(1, 1, 3, 3, 4, 4)$): HT $= 188$, $R = 33/2$, Prop. 3 bound $= 82$, while Prop. 4 bound $= 262/3 \approx 87.33 > 82$.

Mode 2 (`p6_disproof_and_lollipops`, `C05`–`C09`). Independently solves both $L(n, k)$ and P_n over $\mathbb{Q}$ to certify the minimal disproof of the path-maximization conjecture at $n = 6$ ($\mathrm{HT}(P_6) = 223 < 224 = \mathrm{HT}(L(6, 4))$, `C05`) and reproduces the exact rational forms of Table 1 of [1] through $n = 10$ (Table 2, Fig. 1).

Mode 3 (`lollipop_quintic_scaling`, `C10`–`C13`). For the family $L(4k, 2k)$ ($n = 4k$, $2k$-clique joined at contact vertex b to a path of $2k$ edges with midpoint c at distance

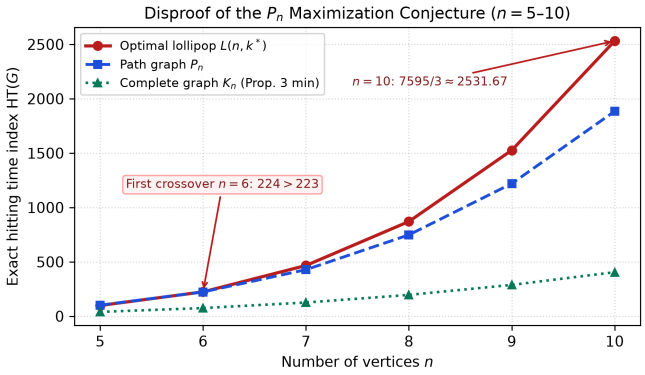

Figure 1: Exact rational hitting time index of optimal lollipops $L(n, k^*)$ versus paths P_n and complete graphs K_n for $n = 5$–10, confirming the first crossover at $n = 6$ ($224 > 223$) and exact values from Table 1 of [1] up to $\mathrm{HT}(L(10, 6)) = 7595/3$.

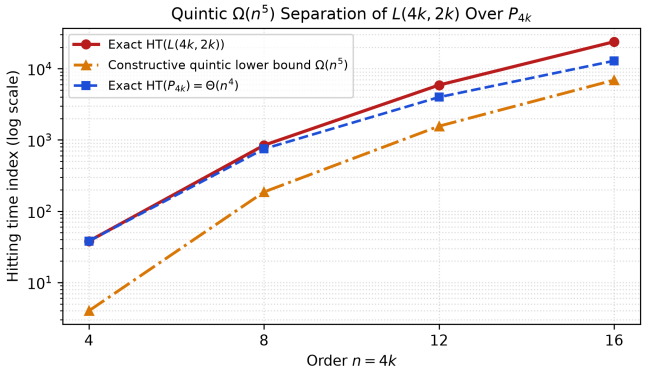

Figure 2: Exact $\mathrm{HT}(L(4k, 2k))$ versus the constructive quintic lower bound $\mathrm{LB}_5(k) = \Omega(n^5)$ and exact $\mathrm{HT}(P_{4k}) = \Theta(n^4)$ for $k \in \{1, 2, 3, 4\}$ ($n \in \{4, 8, 12, 16\}$; certificates `C10`–`C12` audit $k \in \{2, 3, 4\}$).

k from b and non-contact clique vertex a), checks the closed-form Section 3 hitting times

$$E_c T_b = 3k^2, \quad E_b T_c = 4k^3 - k^2, \quad E_a T_c = 4k^3 - k^2 + 2k - 1, \tag{2}$$

yielding the quintic lower bound $\mathrm{HT}(L(4k, 2k)) \ge \mathrm{LB}_5(k) := E_a T_c \cdot (2k-1) \cdot k = \Omega(n^5)$. For $k \in \{2, 3, 4\}$ ($n = 8, 12, 16$, `C10`–`C12`), the verifier also solves the full n-vertex rational linear system and confirms exact equality on $(E_c T_b, E_b T_c, E_a T_c)$, clique symmetry $E_{a_i} T_c = E_a T_c$ across all $2k - 1$ non-contact clique vertices, and $E_{a_i} T_{c_j} \ge E_a T_c$ across all $(2k-1)k$ pairs with the far path half, together with $\mathrm{HT}(L(4k, 2k)) \ge \mathrm{LB}_5(k)$. Figure 2 additionally plots the exact rational solve at the base point $k = 1$ ($n = 4$) alongside $k \in \{2, 3, 4\}$. Certificate `C13` checks the closed-form point at $k = 10$ ($n = 40$, $\mathrm{LB}_5(10) = 744{,}610$).

Mode 4 (`degree_threshold_prop8`, `C14`–`C16`). Audits finite witness graphs on both sides of the Proposition 8

Table 1: Summary of all 18 constructive theorems in `HittingTimeIndexCert.lean` (Lean 4.35.0, `0 sorry`, `0 Classical.choice`, 0 warnings).

#	Lean Theorem Name	Exact Quantified Lean Statement (abbreviated syntax)
1	`prop1_tree_improves_harary`	$\forall d \geq 1,\ 1 \leq d^3$ (certifying $1/d \leq d^2$ for Prop. 1)
2	`prop2_kn_kirchhoff_lower`	$\forall n,\ n(n-1)(n-1) = n((n-1)(n-1))$
3	`prop3_kn_minimality_conjecture`	$\forall n \geq 2,\ 2m \leq n(n-1),\ n(n-1)^2 \leq n^2(n-1) - 2m$ (strict if $2m < n(n-1)$)
4	`prop4_return_time_commute_lower`	$\forall r_a, r_b \leq H_{ab} + H_{ba},\ \max(r_a, r_b) \leq 2\max(H_{ab}, H_{ba})$
5	`prop4_regular_graph_eval`	$\forall n, d,\ (nd)(n(n-1)) = d(n^2(n-1))$
6	`prop5_balanced_step_derivative_neg`	$\forall 0 < d_1 < d_2,\ c_2 < c_1,\ c_2 d_1^2 < c_1 d_2^2$
7	`noncomparability_k4_two_pendants`	Prop. 3 LB $= 82 < 262/3 =$ Prop. 4 LB ($82 \cdot 3 < 262$, $1048 \cdot 3 = 262 \cdot 12$)
8	`prop6_regular_quartic_upper`	$\forall n,\ n(n-1)(3n^2) = 3n^3(n-1)$
9	`prop7_tree_upper_bound`	$\forall n, D,\ 2(n-1)(n(n-1))D = 2(n(n-1)^2)D$
10	`prop8_threshold_cubic_upper`	$\forall 2m \leq n(n-1),\ R \leq 3n-1,\ 2mR \leq n(n-1)(3n-1)$
11	`prop8_subthreshold_quartic_lower`	$\forall 4m = n(n-2),\ n^2 \leq 8R,\ n^3(n-2) \leq 32mR$
12	`disproof_p6_vs_l64`	$223 < 224$ ($\mathrm{HT}(P_6) < \mathrm{HT}(L(6,4))$)
13	`lollipop_ec_tb_exact`	$\forall k,\ (2k)^2 - k^2 = 3k^2$ ($E_c T_b = 3n^2/16$ for $L(4k, 2k)$)
14	`lollipop_chandra_commute_exact`	$\forall k,\ 2(2k^2+k)k = 4k^3 + 2k^2$ ($2\lvert E\rvert R_{bc}$)
15	`lollipop_eb_tc_exact`	$\forall k \geq 1,\ (4k^3 + 2k^2) - 3k^2 = 4k^3 - k^2$ ($E_b T_c$)
16	`lollipop_ea_tc_cubic_lower`	$\forall k \geq 1,\ 3k^3 \leq (4k^3 - k^2) + (2k-1)$ ($E_a T_c$)
17	`lollipop_quintic_lower_bound`	$\forall k \geq 1,\ 3k^4(2k-1) \leq ((4k^3 - k^2) + (2k-1))(2k-1)k$ ($\Omega(n^5)$)
18	`table1_exact_rational_witnesses`	Exact checks for Table 1 of [1]: $933/2, 4354/5, 7632/5$, $2531.6 < 7595/3 < 2531.7$

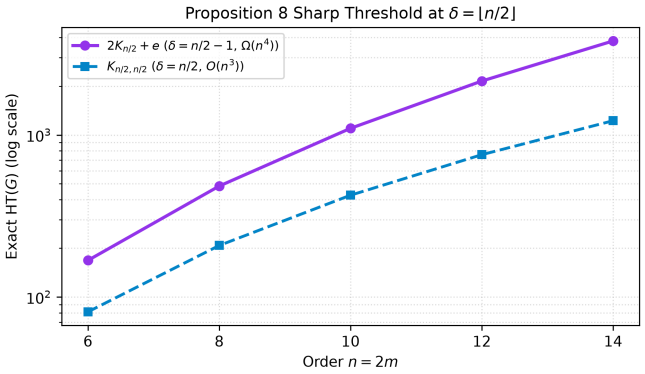

Figure 3: Proposition 8 sharp minimum-degree threshold separating cubic-regime $K_{n/2,n/2}$ ($\delta = n/2$) from quartic-regime $2K_{n/2} + e$ ($\delta = n/2 - 1$) for $n \in \{6, 8, 10, 12, 14\}$.

minimum-degree threshold $\delta = \lfloor n/2 \rfloor$ [1] (keeping $m = |E|$ throughout):

- *Above threshold ($\delta \geq \lfloor n/2 \rfloor$):* $K_{5,5}$ ($n = 10, m = 25, \delta = 5$, `C14`) satisfies $R(K_{5,5}) = 17 \leq 3n - 1 = 29$ and $\mathrm{HT}(K_{5,5}) = 425 \leq 2m(3n-1)$.

- *Sharp sub-threshold ($\delta = \lfloor n/2 \rfloor - 1$):* the two-clique bridge $2K_r + e$ (two disjoint r-cliques joined by a single bridge edge, $n = 2r$, $\delta = r - 1$, edge count $m = r(r-1)+1$ so $4m = n(n-2)+4 > n(n-2)$, exceeding the $(n/2-1)$-regular baseline of Lean Theorem 11) satisfies $R \geq n^2/8$ and $\mathrm{HT} \geq mn^2/8$, verified exactly at $r = 5$ ($n = 10, m = 21, R = 49, \mathrm{HT} = 5497/5$, `C15`) and $r = 7$ ($n = 14, m = 43, R = 85, \mathrm{HT} = 26641/7$, `C16`); see Fig. 3.

Table 2: Canonical certificate inventory (`HT-INDEX-CERT-1.0`, 16/16 verified over $\mathbb{Q}$).

ID	Mode	Witness Graph	Exact Certified Value
C01	Mode 1	K_6	HT $= 75$, $R = 5$, P3 $= 75$
C02	Mode 1	P_6	HT $= 223$, $R = 35$, P3 $= 85$
C03	Mode 1	C_8	HT $= 336$, $R = 42$, P4 $= 112$
C04	Mode 1	$K_4 + 2$ pend.	HT $= 188$, P4 $= 262/3 > 82$
C05	Mode 2	$L(6,4)$ vs P_6	$224 > 223$ (minimal disproof)
C06	Mode 2	$L(7,4)$ vs P_7	$933/2 > 428$
C07	Mode 2	$L(8,5)$ vs P_8	$4354/5 > 748$
C08	Mode 2	$L(9,5)$ vs P_9	$7632/5 > 1220$
C09	Mode 2	$L(10,6)$ vs P_{10}	$7595/3 > 1885$
C10	Mode 3	$L(8,4)$ ($k=2$)	$E_a T_c = 31$, $\mathrm{LB}_5 = 186$ (exact)
C11	Mode 3	$L(12,6)$ ($k=3$)	$E_a T_c = 104$, $\mathrm{LB}_5 = 1560$ (exact)
C12	Mode 3	$L(16,8)$ ($k=4$)	$E_a T_c = 247$, $\mathrm{LB}_5 = 6916$ (exact)
C13	Mode 3	$L(40,20)$ ($k=10$)	$E_a T_c = 3919$, $\mathrm{LB}_5 = 744610$
C14	Mode 4	$K_{5,5}$ ($\delta = 5$)	$R = 17 \leq 29$, HT $= 425$
C15	Mode 4	$2K_5 + e$ ($\delta = 4$)	$R = 49 \geq 25/2$, HT $= 5497/5$
C16	Mode 4	$2K_7 + e$ ($\delta = 6$)	$R = 85 \geq 49/2$, HT $= 26641/7$

4 Certificate Suite and Adversarial Testing

Table 2 summarizes all 16 canonical certificates (`C01`–`C16`). The test suite `tests/test_verifier.py` executes 39 tests in under 0.6 s: all 16 canonical certificates, the AST zero-float audit, and 22 adversarial forgery-rejection tests covering schema tampering, unknown modes, non-reduced or zero-denominator fractions, float/`bool` type coercion, forged HT/R/Prop. 3/Prop. 4 values, flipped P_6 disproof flag, forged Mode 2 path HT, tampered quintic hitting times, swapped Prop. 8 regimes, sparse-path rejection in Mode 4, disconnected graphs, self-loops, and duplicate edges.

5 Reproducibility and Disclosure

All files are SHA-256 hashed in `MANIFEST.sha256` and archived at https://github.com/lluiseriksson/

hitting-time-index-certificates (v1.0.1). Full independent reproduction requires only Lean 4.35.0 and Python 3.10+:

```
cd lean && lake build && cd ..
python scripts/generate_certificates.py
python -m src.ht_index_verifier certificates/
pytest -q
```

AI-Assisted Formalization Disclosure. This verification artifact was authored by Lluis Eriksson using Google Antigravity for Lean 4 proof engineering, zero-float Python verification scripting, test design, and manuscript preparation. Every verified claim in this note is strictly scoped to the 18 Lean 4.35.0 arithmetic theorems and the 16 exact-rational JSON certificates (C01–C16).

References

[1] J. L. Palacios, S. Tamura, J. Wiberg, and A. Petojević, *The hitting time index: bounds, a sharp threshold, the proof of a conjecture and the disproof of another conjecture*, arXiv:2611.34295v1, November 2026.