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Finite Free Root-Gap Dynamics: Universal Marcel Riesz Monotonicity, Sharp Degree-3-to-4 Euclidean and Hyperbolic Pythagorean Transitions, and Exact Heat-Flow Compression

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Abstract

We investigate the evolution of the *minimum root spacing* $\mathrm{gap}(p) = \min_{1\le i<d}(\lambda_{i+1}(p) - \lambda_i(p))$ and the *minimum root ratio* $\mathrm{rat}(p) = \min_{1\le i<d}(\lambda_{i+1}(p)/\lambda_i(p))$ of real-rooted degree-d polynomials under the finite free additive convolution $\boxplus_d$ and finite free multiplicative convolution $\boxtimes_d$ of Marcus, Spielman, and Srivastava. While the finite free entropy power inequality controls the *geometric mean* of all pairwise gaps and holds superadditively in every degree, we show that the *minimum* gap exhibits a sharp low-degree-to-high-degree phase transition at $d = 3 \to 4$ in both the additive (Euclidean) and multiplicative (hyperbolic) worlds:

1. **Universal Finite Free Marcel Riesz Monotonicity:** For all $d \ge 2$, $\mathrm{gap}(p \boxplus_d q) \ge \max(\mathrm{gap}(p), \mathrm{gap}(q))$ on $\mathcal{P}_d^{\mathbb{R}}$ and $\mathrm{rat}(p \boxtimes_d q) \ge \max(\mathrm{rat}(p), \mathrm{rat}(q))$ on $\mathcal{P}_d^{\mathbb{R}^{>0}}$, yielding a universal $\frac{1}{2}$-Pythagorean floor.
2. **Exact Degree-$2,3$ Euclidean and Hyperbolic Laws:** For $d = 2, 3$, additive convolution obeys exact Pythagorean superadditivity $\mathrm{gap}(p \boxplus_3 q)^2 \ge \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2$ (with equality if and only if both factors are symmetric or one is a point mass). Multiplicatively, $d = 2$ obeys the exact hyperbolic Pythagorean identity $\cosh(\frac{1}{2}\log \mathrm{rat}(p \boxtimes_2 q)) = \cosh(\frac{1}{2}\log \mathrm{rat}\, p)\cosh(\frac{1}{2}\log \mathrm{rat}\, q)$, and $d = 3$ obeys the strictly stronger exact supermultiplicative law $\frac{1}{3}\mu(p \boxtimes_3 q) \ge \frac{1}{3}\mu(p) \cdot \frac{1}{3}\mu(q)$ for $\mu(p) = \mathrm{rat}(p) + 1 + \mathrm{rat}(p)^{-1} = 4\cosh^2(\frac{1}{2}\log \mathrm{rat}\, p) - 1$, proved via a deformed cubic discriminant factorization and a three-term semigroup potential $\Phi(\alpha, \beta)$ on $(0, 1)^2$.
3. **Exact Heat-Flow Expansion and Universal $d \ge 4$ Failure:** For the finite free heat flow $p_T = p \boxplus_d \mathrm{He}_d^{(T)}$ driven by deterministic Dyson dynamics, we prove the globally optimal bound $\mathrm{gap}(p_T)^2 \ge \mathrm{gap}(p)^2 + \kappa_d\, \mathrm{gap}(\mathrm{He}_d^{(T)})^2$ for all $d \ge 2$ and $T \ge 0$, where $\kappa_d = c_d^{\mathrm{heat}}/\,\mathrm{gap}(\mathrm{He}_d)^2$ with $c_d^{\mathrm{heat}} = 8/d$ (d even) or $8d/(d^2-1)$ (d odd), satisfying $\kappa_2 = \kappa_3 = 1$, $\kappa_4 = \frac{3+\sqrt{6}}{6}$, $\kappa_5 = \frac{5+\sqrt{10}}{9}$, and $\kappa_d \to \frac{8}{\pi^2} \approx 0.8106$. Sturm–Liouville comparison proves $\kappa_d < 1$ for *every* $d \ge 4$, establishing universal failure of both additive and (via an infinitesimal limit lemma) multiplicative Pythagorean superadditivity for all $d \ge 4$, complemented by closed-form self-convolution counterexamples $5 - \frac{7}{\sqrt{3}}$ ($d = 4$) and $\frac{10-\sqrt{38}}{4}$ ($d = 5$).
4. **Voiculescu R-Transform and the Basel Constant $\pi^2/12$:** For the k-fold free additive convolution $U^{\boxplus k}$ of the uniform measure $U = \mathrm{Unif}[-1/2, 1/2]$, we prove via global analytic continuation and the holomorphic inverse function theorem that $U^{\boxplus k}$ is real-analytic near 0 with exact central density $\rho_k(0) = \frac{2u_k}{\pi}$, where $u_k \in (0, \pi/2)$ solves $u_k \cot u_k = \frac{k-1}{k}$. Its inverse-density-squared ratio $\mathcal{R}_k := \frac{1}{k\rho_k(0)^2} = \frac{\pi^2}{4ku_k^2}$ descends

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strictly from $\frac{\pi^2}{8u_2^2} \approx 0.9081$ ($k = 2$) to the free-CLT Basel limit $\frac{\pi^2}{12} \approx 0.8225$ ($k \to \infty$), matching numerical high-degree self-convolution gap scaling.

The exact algebraic identities, cubic semigroup contraction, Dyson telescoping bounds, counterexamples, and universal Sturm inequalities are formally certified in Lean 4.36.0 (16 kernel-verified theorems, zero `sorry`, zero custom axioms).

1 Introduction

Let $\mathcal{P}_d^{\mathbb{R}}$ denote the set of monic real-rooted polynomials of degree $d \geq 2$, and let $\mathcal{P}_d^{\mathbb{R}>0} \subset \mathcal{P}_d^{\mathbb{R}}$ denote those whose roots are all strictly positive. Writing the ordered roots of $p \in \mathcal{P}_d^{\mathbb{R}}$ as $\lambda_1(p) \leq \lambda_2(p) \leq \cdots \leq \lambda_d(p)$, we study the two most fundamental hard-core spectral separation invariants:

$$\mathrm{gap}(p) := \min_{1 \leq i < d} \big(\lambda_{i+1}(p) - \lambda_i(p)\big) \geq 0, \qquad \mathrm{rat}(p) := \min_{1 \leq i < d} \frac{\lambda_{i+1}(p)}{\lambda_i(p)} \geq 1 \quad (p \in \mathcal{P}_d^{\mathbb{R}>0}). \tag{1}$$

For $p(x) = \sum_{k=0}^{d} (-1)^k a_k x^{d-k}$ and $q(x) = \sum_{k=0}^{d} (-1)^k b_k x^{d-k}$ (with $a_0 = b_0 = 1$), the *finite free additive convolution* $\boxplus_d$ and *finite free multiplicative convolution* $\boxtimes_d$, introduced classically by Walsh [14] and Szegő [10] and revitalized as finite-dimensional models of Voiculescu's free probability by Marcus, Spielman, and Srivastava [7, 6, 1], are defined by

$$(p \boxplus_d q)(x) := \sum_{k=0}^{d} (-1)^k \left(\sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!\,(d-k)!}\, a_i b_j \right) x^{d-k}, \tag{2}$$

$$(p \boxtimes_d q)(x) := \sum_{k=0}^{d} (-1)^k \frac{a_k b_k}{\binom{d}{k}}\, x^{d-k}. \tag{3}$$

By [7], $\mathcal{P}_d^{\mathbb{R}}$ is closed under $\boxplus_d$ and $\mathcal{P}_d^{\mathbb{R}>0}$ is closed under $\boxtimes_d$; moreover, for diagonal matrices $A = \mathrm{diag}(\lambda_1(p), \ldots, \lambda_d(p))$ and $B = \mathrm{diag}(\lambda_1(q), \ldots, \lambda_d(q))$ and Haar-random $Q \in O(d)$,

$$(p \boxplus_d q)(x) = \mathbb{E}_Q \det(xI - A - QBQ^\top), \qquad (p \boxtimes_d q)(x) = \mathbb{E}_Q \det(xI - AQBQ^\top). \tag{4}$$

Recent work by Gribinski [4] and Garza-Vargas, Srivastava, and Stier [3] established finite free analogues of Stam's inequality and the Shannon–Voiculescu entropy power inequality: the finite free entropy power $N_d(p) := |\operatorname{disc}(p)|^{\frac{2}{d(d-1)}} = \big(\prod_{i<j}(\lambda_j - \lambda_i)^2\big)^{\frac{2}{d(d-1)}}$ satisfies

$$N_d(p \boxplus_d q) \geq N_d(p) + N_d(q) \qquad \forall\, d \geq 2,\ \forall\, p, q \in \mathcal{P}_d^{\mathbb{R}}.$$

That inequality governs the *geometric mean* of all $\binom{d}{2}$ pairwise squared root gaps. It is natural to ask whether the *worst-case* nearest-neighbor separation $\mathrm{gap}(p)^2$ enjoys the same Euclidean superadditivity $\mathrm{gap}(p \boxplus_d q)^2 \geq \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2$, and what geometric law governs the multiplicative ratio $\mathrm{rat}(p \boxtimes_d q)$ on $\mathcal{P}_d^{\mathbb{R}>0}$.

We answer both questions completely. Contrary to what the universal discriminant inequality might suggest, the minimum root gap obeys a sharp structural dichotomy between $d \leq 3$ and $d \geq 4$, summarized in Table 1.

Organization. Section 2 proves universal Marcel Riesz monotonicity for both $\boxplus_d$ and $\boxtimes_d$. Section 3 establishes the exact low-degree Euclidean and hyperbolic laws for $d = 2, 3$ and characterizes all equality cases. Section 4 proves the sharp finite free heat-flow expansion theorem via deterministic Dyson dynamics and derives universal $d \geq 4$ failure in both convolutions via Sturm–Liouville comparison and an infinitesimal multiplicative-to-additive limit lemma. Section 5 derives the exact macroscopic R-transform compression law and the Basel limit $\pi^2/12$. Section 6 summarizes the Lean 4.36.0 formal certification, reproducibility digests, and open extremal problems.

Regime	Additive $\boxplus_d$ (Euclidean)	Multiplicative $\boxtimes_d$ (Hyperbolic)	Sharpness / Mechanism
All $d \geq 2$	$\mathrm{gap}(p \boxplus_d q) \geq \max(\mathrm{gap}\, p, \mathrm{gap}\, q)$	$\mathrm{rat}(p \boxtimes_d q) \geq \max(\mathrm{rat}\, p, \mathrm{rat}\, q)$	Obreschkoff + Walsh/Szegő (Thm. 2.1)
$d = 2$	$\mathrm{gap}^2(p \boxplus_2 q) = \mathrm{gap}^2 p + \mathrm{gap}^2 q$	$\cosh c = \cosh a \cosh b$	Exact identities (Thms. 3.1, 3.2)
$d = 3$	$\mathrm{gap}^2(p \boxplus_3 q) \geq \mathrm{gap}^2 p + \mathrm{gap}^2 q$	$\frac{1}{3}\mu(p \boxtimes_3 q) \geq \frac{1}{3}\mu(p)\frac{1}{3}\mu(q)$	Viète / Semigroup Φ (Thms. 3.1, 3.2)
Heat flow $\forall d$	$\mathrm{gap}^2(p_T) \geq \mathrm{gap}^2 p + \kappa_d \mathrm{gap}^2(\mathrm{He}_d^{(T)})$	Fixed-d limit $\boxtimes_d \to \boxplus_d$ ($\varepsilon \to 0^+$)	Dyson telescoping; $\kappa_d \to \frac{8}{\pi^2}$ (Thm. 4.1, Lem. 4.2)
All $d \geq 4$	gap^2 superadditivity **fails**	$\cosh c$, $\mu/3$, $(\log \mathrm{rat})^2$ **fail**	Sturm $\kappa_d < 1$; $5 - \frac{7}{\sqrt{3}}$ (Thm. 4.3)
Free limit ($U^{\boxplus k}$)	Exact $\rho_k(0) = \frac{2u_k}{\pi}$, $\mathcal{R}_k = \frac{\pi^2}{4ku_k^2} \to \frac{\pi^2}{12}$	Heuristic $d \to \infty$ gap scaling (Conj. 6.1)	Voiculescu R-transform (Thm. 5.1)

Table 1: Summary of the main results. Here $a = \frac{1}{2}\log \mathrm{rat}(p)$, $b = \frac{1}{2}\log \mathrm{rat}(q)$, $c = \frac{1}{2}\log \mathrm{rat}(p \boxtimes_d q)$, and $\mu(p) = \mathrm{rat}(p) + 1 + \mathrm{rat}(p)^{-1} = 4\cosh^2 a - 1$.

2 Universal Finite Free Marcel Riesz Monotonicity

A classical theorem of Marcel Riesz [9] states that differentiation never decreases the minimum root gap of a real-rooted polynomial: $\mathrm{gap}(p') \geq \mathrm{gap}(p)$. Because finite free convolution with any $q \in \mathcal{P}_d^{\mathbb{R}}$ acts as a finite-order differential operator preserving real-rootedness, it obeys an exact finite free generalization of Riesz's theorem in both the additive and multiplicative settings.

Theorem 2.1 (Universal Finite Free Marcel Riesz Monotonicity)**.** *Let $d \geq 2$.*

(a) ***Additive monotonicity:*** *For all $p, q \in \mathcal{P}_d^{\mathbb{R}}$,*

$$\mathrm{gap}(p \boxplus_d q) \;\geq\; \max\big(\mathrm{gap}(p),\ \mathrm{gap}(q)\big). \tag{5}$$

In particular, $\mathrm{gap}(p \boxplus_d q)^2 \geq \frac{1}{2}\big(\mathrm{gap}(p)^2 + \mathrm{gap}(q)^2\big)$ *universally for all $d \geq 2$.*

(b) ***Multiplicative monotonicity:*** *For all $p, q \in \mathcal{P}_d^{\mathbb{R}>0}$,*

$$\mathrm{rat}(p \boxtimes_d q) \;\geq\; \max\big(\mathrm{rat}(p),\ \mathrm{rat}(q)\big). \tag{6}$$

Proof. We recall Obreschkoff's Theorem [8]: two real polynomials f, g of degrees d with real roots have weakly interlacing roots (every closed interval between consecutive roots of one contains at least one root of the other, counting multiplicities) if and only if every nontrivial real linear combination $\alpha f + \beta g$ ($\alpha, \beta \in \mathbb{R}$) has only real roots.

Proof of (a). By commutativity of $\boxplus_d$, it suffices to prove $\mathrm{gap}(p \boxplus_d q) \geq \delta := \mathrm{gap}(p)$. The case $\delta = 0$ is trivial, so assume $\delta > 0$. Then the ordered roots of p satisfy $\lambda_1 < \lambda_1 + \delta \leq \lambda_2 < \lambda_2 + \delta \leq \cdots \leq \lambda_d < \lambda_d + \delta$, meaning that $p(x)$ and $p(x - \delta)$ weakly interlace. By Obreschkoff's Theorem, $f_{\alpha,\beta}(x) := \alpha p(x) + \beta p(x - \delta)$ has only real roots for every $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Extend $T_q(f) := f \boxplus_d q$ linearly to the space $\mathbb{R}_{\leq d}[x]$ of polynomials of degree at most d via the differential representation of (2):

$$(T_q f)(x) \;=\; \sum_{j=0}^{d} \frac{(-1)^j (d-j)!}{d!}\, b_j\, f^{(j)}(x), \tag{7}$$

where $q(x) = \sum_{j=0}^{d} (-1)^j b_j x^{d-j}$. Because T_q is a constant-coefficient polynomial in $\frac{d}{dx}$, it commutes with the shift operator $S_\delta f(x) := f(x - \delta)$, so

$$T_q(f_{\alpha,\beta})(x) \;=\; \alpha(p \boxplus_d q)(x) + \beta(p \boxplus_d q)(x - \delta).$$

Whenever $\alpha + \beta \neq 0$, the polynomial $f_{\alpha,\beta}$ has exact degree d and all real roots; by the Walsh–Marcus–Spielman–Srivastava theorem [14, 7] (scaling by $(\alpha + \beta)^{-1}$), $T_q(f_{\alpha,\beta})$ has degree d and all real roots. For the degree-dropping hyperplane $\alpha + \beta = 0$ ($\beta = -\alpha \neq 0$), we write

$$T_q\big(p(x) - p(x - \delta)\big) \;=\; \lim_{\varepsilon \to 0^+} T_q\big((1 + \varepsilon)p(x) - p(x - \delta)\big),$$

a limit of real-rooted degree-d polynomials whose leading coefficient vanishes as $\varepsilon \to 0^+$ while the degree-$(d-1)$ coefficient equals $d\delta \neq 0$. By Hurwitz's theorem, all $d-1$ roots of $T_q(p(x)-p(x-\delta))$ are real. Thus every nontrivial real linear combination of $(p \boxplus_d q)(x)$ and $(p \boxplus_d q)(x-\delta)$ is real-rooted. By the converse of Obreschkoff's Theorem, the ordered roots $\mu_1 \leq \cdots \leq \mu_d$ of $p \boxplus_d q$ weakly interlace with $\mu_1 + \delta \leq \cdots \leq \mu_d + \delta$. Since $\mu_d < \mu_d + \delta$, the interlacing must have orientation $\mu_1 < \mu_1 + \delta \leq \mu_2 < \cdots \leq \mu_d < \mu_d + \delta$, giving $\mu_{i+1} - \mu_i \geq \delta$ for all $1 \leq i < d$. The $\frac{1}{2}$-Pythagorean floor follows immediately from $\max(x,y)^2 \geq \frac{1}{2}(x^2+y^2)$.

Proof of (b). Again by commutativity, assume $\rho := \mathrm{rat}(p) > 1$. The positive roots of p satisfy $0 < \lambda_1 < \rho\lambda_1 \leq \lambda_2 < \cdots \leq \lambda_d < \rho\lambda_d$, so $p(x)$ and $p(x/\rho)$ weakly interlace, and $g_{\alpha,\beta}(x) := \alpha p(x) + \beta p(x/\rho)$ has only real roots (possibly of mixed sign) for all $(\alpha,\beta) \neq (0,0)$. Define the linear operator $M_q : \mathbb{R}_{\leq d}[x] \to \mathbb{R}_{\leq d}[x]$ by

$$M_q(x^m) := \frac{b_{d-m}}{\binom{d}{m}}\, x^m \qquad (0 \leq m \leq d),$$

so that $M_q((-1)^k a_k x^{d-k}) = (-1)^k \frac{a_k b_k}{\binom{d}{k}} x^{d-k}$, i.e., $M_q(p) = p \boxtimes_d q$, and M_q commutes with every dilation $D_\rho f(x) := f(x/\rho)$. Because $q \in \mathcal{P}_d^{\mathbb{R}>0}$ has all positive roots, the associated polynomial $Q(x) := \sum_{k=0}^d b_k x^{d-k} = (-1)^d q(-x)$ has all negative (hence same-sign) roots. Therefore Szegő's Composition Theorem [10] (see also [2, 7]) guarantees that $M_q(f)$ has all real roots for *every* degree-d polynomial f whose roots are all real (regardless of sign). Hence for all $\alpha + \beta\rho^{-d} \neq 0$, $M_q(g_{\alpha,\beta})(x) = \alpha(p \boxtimes_d q)(x) + \beta(p \boxtimes_d q)(x/\rho)$ has degree d and all real roots. For $\alpha + \beta\rho^{-d} = 0$, the coefficient of x^{d-1} in $M_q(p(x) - \rho^d p(x/\rho)) = (p \boxtimes_d q)(x) - \rho^d (p \boxtimes_d q)(x/\rho)$ is $-\frac{a_1 b_1}{d}(1-\rho) \neq 0$ (since $a_1, b_1 > 0$ and $\rho > 1$), so Hurwitz's theorem again yields real-rootedness of $M_q(g_{\alpha,\beta})$. By Obreschkoff's Theorem, the positive roots $0 < \mu_1 \leq \cdots \leq \mu_d$ of $p \boxtimes_d q$ weakly interlace with $\rho\mu_1 \leq \cdots \leq \rho\mu_d$; since $\mu_d < \rho\mu_d$, we obtain $\rho\mu_i \leq \mu_{i+1}$ for all $1 \leq i < d$, i.e., $\mathrm{rat}(p \boxtimes_d q) \geq \rho$. □

3 Exact Low-Degree Laws: Euclidean and Hyperbolic Superadditivity ($d \leq 3$)

For degrees $d = 2$ and $d = 3$, the minimum root separation satisfies sharp superadditive laws far stronger than Theorem 2.1. In the additive setting, the geometry is Euclidean; in the multiplicative setting, it is hyperbolic.

Theorem 3.1 (Exact Additive Pythagorean Laws for $d = 2, 3$)**.**

(a) ***Degree*** *2: For all $p, q \in \mathcal{P}_2^{\mathbb{R}}$, $\mathrm{gap}(p \boxplus_2 q)^2 = \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2$.*

(b) ***Degree*** *3: For all $p, q \in \mathcal{P}_3^{\mathbb{R}}$,*

$$\mathrm{gap}(p \boxplus_3 q)^2 \;\geq\; \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2. \tag{8}$$

Equality holds in (8) if and only if either (i) both p and q are symmetric (i.e., affine images of $\mathrm{He}_3(x) = x^3 - 3x$, equivalently arithmetic progressions), or (ii) at least one of p, q is a point mass $(x-c)^3$.

Proof. Part (a) is immediate: writing $p(x) = x^2 - e_1 x + e_2$ and $q(x) = x^2 - f_1 x + f_2$, we have $(p \boxplus_2 q)(x) = x^2 - (e_1+f_1)x + (e_2 + f_2 + \frac{1}{2}e_1 f_1)$, whose discriminant is $(e_1+f_1)^2 - 4(e_2+f_2+\frac{1}{2}e_1 f_1) = (e_1^2 - 4e_2) + (f_1^2 - 4f_2) = \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2$.

For part (b), since gap is translation-invariant and $\boxplus_3$ is translation-equivariant, we center both cubics to have root sum zero:

$$p(x) = x^3 - 3A_1 x + 2B_1, \qquad q(x) = x^3 - 3A_2 x + 2B_2,$$

with $A_i \geq 0$ and $B_i^2 \leq A_i^3$ (real-rootedness). By (2), centered cubic convolution is exact vector addition:

$$(p \boxplus_3 q)(x) \;=\; x^3 - 3(A_1 + A_2)x + 2(B_1 + B_2).$$

We claim that for any centered $r(x) = x^3 - 3Ax + 2B \in \mathcal{P}_3^{\mathbb{R}}$ and any $g \geq 0$,

$$\operatorname{gap}(r) \geq g \quad\Longleftrightarrow\quad A \geq \frac{g^2}{3} \quad \text{and} \quad B^2 \leq \Big(A - \frac{g^2}{3}\Big)^2 \Big(A - \frac{g^2}{12}\Big). \tag{9}$$

Indeed, if $A = 0$ then $B = 0$ and both sides are equivalent to $g = 0$. If $A > 0$, Viète's trigonometric representation gives the ordered roots of r as $\lambda_k = 2\sqrt{A}\cos\big(\theta + \frac{2\pi k}{3}\big)$ for a unique $\theta \in [0, \pi/3]$ satisfying $\cos(3\theta) = -B/A^{3/2}$: explicitly,

$$\lambda_1 = -2\sqrt{A}\cos\Big(\frac{\pi}{3} - \theta\Big), \quad \lambda_2 = -2\sqrt{A}\cos\Big(\frac{\pi}{3} + \theta\Big), \quad \lambda_3 = 2\sqrt{A}\cos\theta.$$

The two adjacent gaps are $2\sqrt{3A}\sin(\frac{\pi}{3} - \theta)$ and $2\sqrt{3A}\sin\theta$, so setting $\phi := \min(\theta, \frac{\pi}{3} - \theta) \in [0, \pi/6]$ we have $\operatorname{gap}(r) = 2\sqrt{3A}\sin\phi$ and $|\cos(3\theta)| = \cos(3\phi) = \sqrt{1 - \sin^2(3\phi)}$. Putting $s := \sin\phi \in [0, 1/2]$, we have $\sin(3\phi) = s(3 - 4s^2)$ and the exact algebraic factorization (certified in Lean 4 as `cubic_viete_factorization`):

$$1 - s^2(3 - 4s^2)^2 \;=\; (1 - s^2)(1 - 4s^2)^2.$$

Hence $B^2/A^3 = \cos^2(3\phi) = (1 - s^2)(1 - 4s^2)^2$, which is strictly decreasing in $s \in [0, 1/2]$ (as both $1 - s^2$ and $(1 - 4s^2)^2$ decrease). Thus $\operatorname{gap}(r) \geq g \iff s \geq \frac{g}{2\sqrt{3A}}$ (requiring $\frac{g}{2\sqrt{3A}} \leq \frac{1}{2}$, i.e., $A \geq g^2/3$), which is equivalent to

$$\frac{B^2}{A^3} \;\leq\; \Big(1 - \frac{g^2}{12A}\Big)\Big(1 - \frac{g^2}{3A}\Big)^2 \iff B^2 \;\leq\; \Big(A - \frac{g^2}{3}\Big)^2\Big(A - \frac{g^2}{12}\Big),$$

proving (9).

Now let $g_1 := \operatorname{gap}(p)$, $g_2 := \operatorname{gap}(q)$, $u_i := A_i - g_i^2/3 \geq 0$, and $v_i := A_i - g_i^2/12 = u_i + g_i^2/4 \geq 0$. By (9), $|B_i| \leq u_i\sqrt{v_i}$. Set $g_* := \sqrt{g_1^2 + g_2^2}$, so $A_1 + A_2 - g_*^2/3 = u_1 + u_2 \geq 0$ and $A_1 + A_2 - g_*^2/12 = v_1 + v_2 \geq 0$. Then

$$\begin{aligned}(u_1 + u_2)^2(v_1 + v_2) - (B_1 + B_2)^2 &\geq (u_1 + u_2)^2(v_1 + v_2) - (u_1\sqrt{v_1} + u_2\sqrt{v_2})^2 \\ &= u_1^2 v_2 + u_2^2 v_1 + 2u_1u_2\big(v_1 + v_2 - \sqrt{v_1v_2}\big) \;\geq\; 0, \end{aligned} \tag{10}$$

since $v_1 + v_2 \geq 2\sqrt{v_1v_2} \geq \sqrt{v_1v_2}$. By (9), $\operatorname{gap}(p \boxplus_3 q) \geq g_* = \sqrt{g_1^2 + g_2^2}$.

For equality: if $g_1 > 0$ and $g_2 > 0$, then $v_1, v_2 > 0$, so (10) vanishes if and only if $u_1 = u_2 = 0$, i.e., $B_1 = B_2 = 0$ (both cubics are symmetric). If $g_1 = 0$ and $g_2 > 0$, then $v_2 > 0$ and $u_1 = v_1 = A_1$, so (10) vanishes if and only if $A_1 = 0$ (so $p(x) = x^3$, a point mass); symmetrically, if $g_1 > 0$ and $g_2 = 0$, then (10) vanishes if and only if $A_2 = 0$ (so $q(x) = x^3$). If $g_1 = g_2 = 0$, then $u_i = v_i = A_i$ and $v_1 + v_2 - \sqrt{v_1v_2} > 0$ whenever $A_1, A_2 > 0$, so (10) vanishes if and only if $A_1 = 0$ or $A_2 = 0$ (a point mass). Conversely, in both cases (i) and (ii), equality holds directly. □

We now turn to the multiplicative convolution $\boxtimes_d$ on $\mathcal{P}_d^{\mathbb{R}_{>0}}$. Defining the half-log-ratio $a(p) := \frac{1}{2}\log\operatorname{rat}(p) \geq 0$ and the cubic ratio invariant

$$\mu(p) \;:=\; \operatorname{rat}(p) + 1 + \operatorname{rat}(p)^{-1} \;=\; 4\cosh^2\big(a(p)\big) - 1 \;\geq\; 3, \tag{11}$$

we discover an exact hyperbolic Pythagorean identity at $d = 2$ and an even stronger supermultiplicative law at $d = 3$.

Theorem 3.2 (Exact Multiplicative Hyperbolic and Cubic μ-Laws for $d = 2, 3$)**.**

(a) ***Degree*** 2 ***(Hyperbolic Pythagorean Identity):*** *For all* $p, q \in \mathcal{P}_2^{\mathbb{R}_{>0}}$,

$$\cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(p \boxtimes_2 q)\Big) \;=\; \cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(p)\Big)\,\cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(q)\Big). \tag{12}$$

In particular, $(\log \mathrm{rat}(p \boxtimes_2 q))^2 \geq (\log \mathrm{rat}(p))^2 + (\log \mathrm{rat}(q))^2$, *with equality if and only if* $\mathrm{rat}(p) = 1$ *or* $\mathrm{rat}(q) = 1$.

(b) ***Degree*** 3 ***(Cubic*** μ***-Supermultiplicativity and Hyperbolic Pythagoras):*** *For all* $p, q \in \mathcal{P}_3^{\mathbb{R}_{>0}}$,

$$\frac{\mu(p \boxtimes_3 q)}{3} \;\geq\; \frac{\mu(p)}{3} \cdot \frac{\mu(q)}{3}. \tag{13}$$

Equality holds in (13) *if and only if either (i) both* p *and* q *are geometric progressions (i.e., monic cubics of the form* $(x - c\rho^{-1})(x - c)(x - c\rho)$ *with* $c > 0$ *and* $\rho \geq 1$*), or (ii) at least one of* p, q *is a point mass* $(x - c)^3$. *Furthermore,* (13) *strictly implies the cubic hyperbolic Pythagorean inequality*

$$\cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(p \boxtimes_3 q)\Big) \;\geq\; \cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(p)\Big)\,\cosh\Big(\tfrac{1}{2}\log \mathrm{rat}(q)\Big), \tag{14}$$

as well as $(\log \mathrm{rat}(p \boxtimes_3 q))^2 \geq (\log \mathrm{rat}(p))^2 + (\log \mathrm{rat}(q))^2$.

Proof. Proof of (a). By dilation invariance of rat, normalize $p(x) = (x-1)(x-\alpha)$ and $q(x) = (x-1)(x-\beta)$ with $\alpha = \mathrm{rat}(p) \geq 1$ and $\beta = \mathrm{rat}(q) \geq 1$. Then $(p \boxtimes_2 q)(x) = x^2 - \frac{(1+\alpha)(1+\beta)}{2}x + \alpha\beta$ has positive roots $0 < \gamma_1 \leq \gamma_2$ with $\gamma_1\gamma_2 = \alpha\beta$ and $\gamma_1 + \gamma_2 = \frac{1}{2}(1+\alpha)(1+\beta)$. Setting $\gamma := \gamma_2/\gamma_1 = \mathrm{rat}(p \boxtimes_2 q)$, we obtain

$$2\cosh\Big(\tfrac{1}{2}\log\gamma\Big) = \sqrt{\gamma} + \frac{1}{\sqrt{\gamma}} = \frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}} = \frac{(1+\alpha)(1+\beta)}{2\sqrt{\alpha\beta}} = 2\cosh\Big(\tfrac{1}{2}\log\alpha\Big)\cosh\Big(\tfrac{1}{2}\log\beta\Big),$$

proving (12). To deduce $(\log\gamma)^2 \geq (\log\alpha)^2 + (\log\beta)^2$, note that for $a, b \geq 0$, comparing Taylor series yields $\cosh(\sqrt{a^2+b^2}) = \sum_{n=0}^{\infty} \frac{(a^2+b^2)^n}{(2n)!} \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{2k}}{(2k)!}\frac{b^{2(n-k)}}{(2n-2k)!} = \cosh(a)\cosh(b)$, because $\binom{2n}{2k} \geq \binom{n}{k}$ for all $0 \leq k \leq n$ (strictly for $n \geq 2$ and $1 \leq k \leq n-1$).

Proof of (b). First verify that (13) implies (14): writing $A = \cosh^2 a(p) \geq 1$, $B = \cosh^2 a(q) \geq 1$, and $C = \cosh^2 a(p \boxtimes_3 q) \geq 1$, (13) reads $4C - 1 \geq \frac{1}{3}(4A-1)(4B-1) = 4AB - 1 + \frac{4}{3}(A-1)(B-1) \geq 4AB - 1$, hence $C \geq AB$, which is (14) (and implies Euclidean log-ratio superadditivity as in part (a)).

Now we prove (13) in four elementary steps. By dilation invariance, normalize every $p \in \mathcal{P}_3^{\mathbb{R}_{>0}}$ so that its root product is 1:

$$p_{u,v}(x) := x^3 - 3ux^2 + 3vx - 1, \qquad u, v \geq 1.$$

Under this normalization, (3) becomes exact componentwise multiplication:

$$p_{u_1,v_1} \boxtimes_3 p_{u_2,v_2} \;=\; p_{u_1u_2,\, v_1v_2}. \tag{15}$$

Step 1 (Deformed cubic discriminant). Fix $\rho \geq 1$ and $\mu := \rho + 1 + \rho^{-1} \geq 3$. A cubic $p_{u,v}$ has a pair of positive roots $(s, \rho s)$ with ratio ρ if and only if $p_{u,v}(s) = 0$ and $p_{u,v}(\rho s) = 0$ share a common root $s > 0$. For $\rho > 1$, eliminating the constant term via $(p_{u,v}(\rho s) - p_{u,v}(s))/(s(\rho-1)) = 0$ and eliminating the cubic term via $(p_{u,v}(\rho s) - \rho^3 p_{u,v}(s))/(\rho - 1) = 0$ yields two quadratics in s:

$$(\rho^2+\rho+1)s^2 - 3u(\rho+1)s + 3v = 0, \qquad 3u\rho^2 s^2 - 3v\rho(\rho+1)s + (\rho^2+\rho+1) = 0.$$

Their Sylvester resultant in s equals $-\rho^4 \mu\, D_\mu(u,v)$, where the *deformed cubic discriminant* is defined by

$$D_\mu(u,v) \;:=\; 81u^2v^2 + 9\mu(\mu+3)uv - 27(\mu+1)(u^3+v^3) - \mu^3. \tag{16}$$

Notice that at $\mu = 3$ ($\rho = 1$), $D_3(u,v) = 81u^2v^2+162uv-108(u^3+v^3)-27 = \Delta(p_{u,v})$ is precisely the ordinary cubic discriminant. Introducing the μ-normalized coordinates $\tilde{u} := 3u/\mu$ and $\tilde{v} := 3v/\mu$, direct algebra (certified in Lean 4 as `deformed_cubic_discriminant_factorization`) gives the fundamental factorization:

$$D_\mu(u,v) \;=\; \mu^3\Big[\mu\big(\tilde{u}^2-\tilde{v}\big)\big(\tilde{v}^2-\tilde{u}\big) - \big(\tilde{u}^3+\tilde{v}^3+1-3\tilde{u}\tilde{v}\big)\Big]. \tag{17}$$

Step 2 (Exact domain characterization). We claim that for every $\rho \geq 1$ ($\mu = \rho + 1 + \rho^{-1}$),

$$\mathrm{rat}(p_{u,v}) \geq \rho \quad\Longleftrightarrow\quad \tilde{u} \geq 1,\;\; \tilde{v} \geq 1,\;\; \tilde{u}^2 \geq \tilde{v},\;\; \tilde{v}^2 \geq \tilde{u},\;\; \text{and}\;\; D_\mu(u,v) \geq 0. \tag{18}$$

Denote the right-hand set in $(\tilde{u},\tilde{v})$ by $\mathcal{K}_\mu$. First, suppose $\mathrm{rat}(p_{u,v}) \geq \rho$. Write the ordered roots as $r_1 = x^{-2/3}y^{-1/3}$, $r_2 = x^{1/3}y^{-1/3}$, $r_3 = x^{1/3}y^{2/3}$ with $x = r_2/r_1 \geq \rho$ and $y = r_3/r_2 \geq \rho$. As $x, y \geq \rho \geq 1$, the function $3u(x,y) = x^{1/3}y^{2/3} + x^{1/3}y^{-1/3} + x^{-2/3}y^{-1/3}$ is strictly increasing in both x and y on $[\rho,\infty)^2$ (indeed, $\frac{\partial(3u)}{\partial x} = \frac{1}{3}x^{-2/3}y^{-1/3}(y+1-2/x) \geq \frac{1}{3}x^{-2/3}y^{-1/3}(2-2/x) \geq 0$, and $\frac{\partial(3u)}{\partial y} = \frac{1}{3}x^{-2/3}y^{-4/3}(2xy-x-1) > 0$ for $xy > 1$). Hence $3u(x,y) \geq 3u(\rho,\rho) = \mu$, giving $\tilde{u} \geq 1$, and by symmetry $v(x,y) = u(y,x)$, giving $\tilde{v} \geq 1$. Moreover, $D_\mu(u,v)$ vanishes only when some root ratio equals ρ (on the boundary $x = \rho$ or $y = \rho$, since $r_3/r_1 = xy \geq \rho^2 > \rho$ for $\rho > 1$), and $D_\mu(u,v) > 0$ as $x = y \to \infty$ (where $\tilde{u} = \tilde{v} \to \infty$ and $D_\mu/\mu^3 \sim \mu\tilde{u}^4 > 0$). By connectedness of $(\rho,\infty)^2$, $D_\mu(u,v) > 0$ whenever $x, y > \rho$, and $D_\mu(u,v) \geq 0$ for $x, y \geq \rho$. Since $\tilde{u}^3+\tilde{v}^3+1-3\tilde{u}\tilde{v} = \frac{1}{2}(\tilde{u}+\tilde{v}+1)\big((\tilde{u}-\tilde{v})^2+(\tilde{u}-1)^2+(\tilde{v}-1)^2\big) \geq 0$ (strictly unless $\tilde{u} = \tilde{v} = 1$), (17) forces $(\tilde{u}^2-\tilde{v})(\tilde{v}^2-\tilde{u}) \geq 0$. Because $\tilde{u}, \tilde{v} \geq 1$ cannot simultaneously satisfy $\tilde{u}^2 < \tilde{v}$ and $\tilde{v}^2 < \tilde{u}$ (which would give $\tilde{u}^4 < \tilde{u}$), we have $\tilde{u}^2 \geq \tilde{v}$ and $\tilde{v}^2 \geq \tilde{u}$, with both inequalities strict whenever $(\tilde{u},\tilde{v}) \neq (1,1)$.

Conversely, suppose $(\tilde{u},\tilde{v}) \in \mathcal{K}_\mu$. If $(\tilde{u},\tilde{v}) = (1,1)$, then $p_{u,v}(x) = (x-\rho^{-1})(x-1)(x-\rho)$ has rat $= \rho$. Otherwise $\tilde{u}^2 > \tilde{v}$ and $\tilde{v}^2 > \tilde{u}$, so the auxiliary variables

$$\alpha := \frac{\tilde{v}}{\tilde{u}^2} \in (0,1), \qquad \beta := \frac{\tilde{u}}{\tilde{v}^2} \in (0,1) \tag{19}$$

are well defined and satisfy $\tilde{u} = (\alpha^2\beta)^{-1/3}$, $\tilde{v} = (\alpha\beta^2)^{-1/3}$. Substituting (19) into (17) transforms $D_\mu(u,v) \geq 0$ into $\Phi(\alpha,\beta) \leq \mu$, where

$$\Phi(\alpha,\beta) \;:=\; \frac{\tilde{u}^3+\tilde{v}^3+1-3\tilde{u}\tilde{v}}{(\tilde{u}^2-\tilde{v})(\tilde{v}^2-\tilde{u})} \;=\; \frac{(\alpha-\beta)^2}{(1-\alpha)(1-\beta)} + \alpha + \beta + \alpha\beta. \tag{20}$$

(Verification of (20): setting $r = 1/\alpha = \tilde{u}^2/\tilde{v}$ and $s = 1/\beta = \tilde{v}^2/\tilde{u}$, we have $\tilde{u}^3 = r^2s$, $\tilde{v}^3 = rs^2$, $\tilde{u}\tilde{v} = rs$, and $(\tilde{u}^2-\tilde{v})(\tilde{v}^2-\tilde{u}) = rs(r-1)(s-1)$, giving $\Phi = \frac{r+s+(rs)^{-1}-3}{(r-1)(s-1)} = \frac{\beta(1-\alpha)}{1-\beta} + \frac{\alpha(1-\beta)}{1-\alpha} + \alpha\beta = \frac{(\alpha-\beta)^2}{(1-\alpha)(1-\beta)} + \alpha + \beta + \alpha\beta$.)

Furthermore, for all $\mu \geq 3$ and $u, v \geq \mu/3$, setting $a = 3u-\mu \geq 0$ and $b = 3v-\mu \geq 0$ yields the exact identity (certified in Lean 4 as `discriminant_dominates_deformed`):

$$\Delta(p_{u,v}) - D_\mu(u,v) \;\geq\; (\mu-3)^3(\mu+1) \;\geq\; 0. \tag{21}$$

Hence if $\mu > 3$ and $(\tilde{u},\tilde{v}) \in \mathcal{K}_\mu$, then $\Delta(p_{u,v}) > 0$, so $p_{u,v}$ has three distinct real roots (all positive by Descartes' rule of signs, since $u, v > 0$). Moreover, for any $(\alpha,\beta) \in (0,1)^2$ with $\Phi(\alpha,\beta) < \mu$ (the interior $\mathcal{K}_\mu^\circ$), the radial ray $\tau \mapsto (\tau\alpha, \tau\beta)$ for $\tau \in (0,1]$ satisfies

$$\Phi(\tau\alpha,\tau\beta) = \frac{\tau}{1-\tau\alpha}\cdot\frac{\tau}{1-\tau\beta}(\alpha-\beta)^2 + \tau\alpha + \tau\beta + \tau^2\alpha\beta,$$

which is strictly increasing in τ! Thus $\mathcal{K}_\mu^\circ$ is *star-shaped* toward $(0,0)$ (see Figure 1), hence connected. Along the ray as $\tau \to 0^+$, $\tilde{u}, \tilde{v} \to \infty$ with $\alpha/\beta = \text{const}$, whence the root ratios $r_2/r_1, r_3/r_2 \to \infty$ exceed ρ. Since $D_\mu > 0$ on $\mathcal{K}_\mu^\circ$, no root ratio can cross ρ along the ray; by

continuity, $\mathrm{rat}(p_{u,v}) > \rho$ on $\mathcal{K}_\mu^\circ$ and $\mathrm{rat}(p_{u,v}) \geq \rho$ on its closure $\mathcal{K}_\mu$. (For $\mu = 3$, $D_3 = \Delta \geq 0$ directly characterizes $\mathcal{P}_3^{\mathbb{R}>0}$.) This proves (18).

Step 3 (Semigroup closure of Φ). We prove that for all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in (0,1)$,

$$\Phi(\alpha_1\alpha_2,\ \beta_1\beta_2) \ <\ \max\big(\Phi(\alpha_1,\beta_1),\ \Phi(\alpha_2,\beta_2)\big). \tag{22}$$

Decompose $\Phi(\alpha,\beta) = Q(\alpha,\beta) + L(\alpha,\beta)$, where $Q(\alpha,\beta) := \frac{(\alpha-\beta)^2}{(1-\alpha)(1-\beta)} = R + R^{-1} - 2$ with $R(\alpha,\beta) := \frac{1-\alpha}{1-\beta} > 0$, and $L(\alpha,\beta) := \alpha+\beta+\alpha\beta$. Clearly $L(\alpha_1\alpha_2, \beta_1\beta_2) < \min(L(\alpha_1,\beta_1), L(\alpha_2,\beta_2))$. For $R_{12} := R(\alpha_1\alpha_2, \beta_1\beta_2)$, let $M := \max(1, R_1, R_2)$ where $R_i = R(\alpha_i,\beta_i)$. We claim that $R_{12} \leq M$ (verified in Lean 4 by `ratio_contraction_upper_int`):

- If $\alpha_1 \leq \beta_1$, then $1-\alpha_1\alpha_2 = (1-\alpha_1)+\alpha_1(1-\alpha_2) \leq M(1-\beta_1)+\beta_1 M(1-\beta_2) = M(1-\beta_1\beta_2)$.
- If $\alpha_2 \leq \beta_2$, the same estimate holds symmetrically with indices $1 \leftrightarrow 2$.
- If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1\alpha_2 > \beta_1\beta_2$, so $R_{12} < 1 \leq M$.

Swapping $\alpha \leftrightarrow \beta$ yields $R_{12}^{-1} \leq \max(1, R_1^{-1}, R_2^{-1})$. Since $x \mapsto x + x^{-1} - 2$ is increasing on $[1,\infty)$ and symmetric under $x \leftrightarrow x^{-1}$, we deduce $Q(\alpha_1\alpha_2, \beta_1\beta_2) \leq \max(Q(\alpha_1,\beta_1), Q(\alpha_2,\beta_2))$. Adding $L(\alpha_1\alpha_2, \beta_1\beta_2) < \min(L_1, L_2)$ gives

$$\Phi(\alpha_1\alpha_2, \beta_1\beta_2) \ <\ \max(Q_1, Q_2) + \min(L_1, L_2) \ \leq\ \max(Q_1 + L_1, Q_2 + L_2),$$

proving (22).

Step 4 (Conclusion and equality characterization). Let $p = p_{u_1,v_1}$ and $q = p_{u_2,v_2}$ have $\mu_1 = \mu(p) \geq 3$ and $\mu_2 = \mu(q) \geq 3$, and set $\mu_* := \frac{1}{3}\mu_1\mu_2 \geq \max(\mu_1,\mu_2) \geq 3$. Under the scaling $\tilde{u}_i = 3u_i/\mu_i$, $\tilde{v}_i = 3v_i/\mu_i$, the product polynomial $p \boxtimes_3 q = p_{u_1u_2, v_1v_2}$ has μ_*-normalized coordinates

$$\tilde{u}_* = \frac{3u_1u_2}{\mu_*} = \frac{3u_1}{\mu_1}\cdot\frac{3u_2}{\mu_2} = \tilde{u}_1\tilde{u}_2, \qquad \tilde{v}_* = \tilde{v}_1\tilde{v}_2,$$

and hence $\alpha_* = \alpha_1\alpha_2$ and $\beta_* = \beta_1\beta_2$!

- If $(\tilde{u}_1, \tilde{v}_1) = (1,1)$ and $(\tilde{u}_2, \tilde{v}_2) = (1,1)$ (both are geometric progressions), then $(\tilde{u}_*, \tilde{v}_*) = (1,1) \in \mathcal{K}_{\mu_*}$, giving exact equality $\mu(p \boxtimes_3 q) = \mu_*$.
- If one factor is a point mass (say $p(x) = (x-1)^3$, so $\mu_1 = 3$ and $\tilde{u}_1 = \tilde{v}_1 = 1$), then $p \boxtimes_3 q = q$, giving equality $\mu(p \boxtimes_3 q) = \mu_2 = \mu_*$.
- If one factor is a non-degenerate geometric progression (say $\tilde{u}_2 = \tilde{v}_2 = 1$, $\mu_2 > 3$) and the other is not ($(\tilde{u}_1, \tilde{v}_1) \neq (1,1)$), then $(\tilde{u}_*, \tilde{v}_*) = (\tilde{u}_1, \tilde{v}_1)$ satisfies $\Phi(\alpha_1,\beta_1) \leq \mu_1 < \frac{1}{3}\mu_1\mu_2 = \mu_*$ (strictly, even if $\mu_1 = 3$), so $(\tilde{u}_*, \tilde{v}_*) \in \mathcal{K}_{\mu_*}^\circ$ and $\mu(p \boxtimes_3 q) > \mu_*$ strictly.
- If neither factor is a geometric progression ($(\tilde{u}_i, \tilde{v}_i) \neq (1,1)$ for $i = 1,2$), then $\alpha_i, \beta_i \in (0,1)$, so (22) gives $\Phi(\alpha_1\alpha_2, \beta_1\beta_2) < \max(\Phi_1, \Phi_2) \leq \max(\mu_1,\mu_2) \leq \mu_*$, placing $(\tilde{u}_*, \tilde{v}_*) \in \mathcal{K}_{\mu_*}^\circ$ and yielding $\mu(p \boxtimes_3 q) > \mu_*$ strictly.

This completes the proof of Theorem 3.2. □

4 Exact Finite Free Heat-Flow Expansion and Universal $d \geq 4$ Failure

Why does Euclidean Pythagorean superadditivity break abruptly at $d = 4$? The deepest explanation comes from the *finite free heat flow*. Let $\mathrm{He}_d(x) := \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \frac{d!}{(d-2k)!\,k!\,2^k} x^{d-2k}$ be the monic probabilists' Hermite polynomial of degree d, and for $T > 0$ let $\mathrm{He}_d^{(T)}(x) :=$

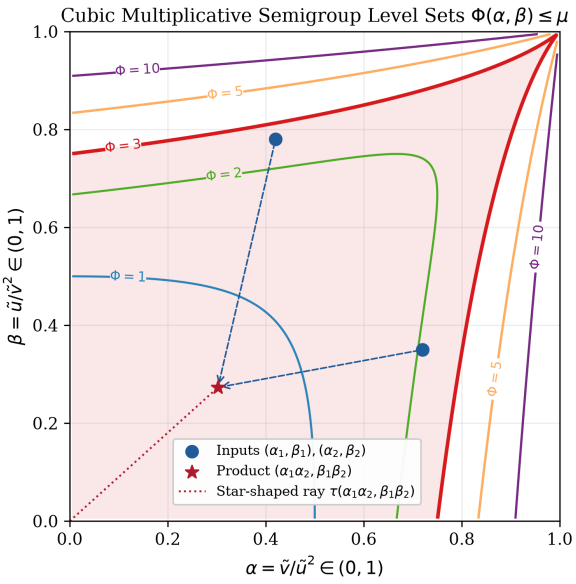


Figure 1: Level sets of the cubic multiplicative semigroup potential $\Phi(\alpha,\beta) = \frac{(\alpha-\beta)^2}{(1-\alpha)(1-\beta)} + \alpha + \beta + \alpha\beta$ on $(0,1)^2$. Every sublevel set $\mathcal{K}_\mu = \{\Phi \le \mu\}$ $(\mu \ge 3)$ is star-shaped toward $(0,0)$ and strictly closed under componentwise multiplication $(\alpha_1,\beta_1)\odot(\alpha_2,\beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2)$.

$T^{d/2}\mathrm{He}_d(x/\sqrt{T})$ (with $\mathrm{He}_d^{(0)}(x) = x^d$). By (7), convolution with $\mathrm{He}_d^{(T)}$ solves the backward heat equation [12, 5]:

$$p_T(x) \;:=\; \big(p \boxplus_d \mathrm{He}_d^{(T)}\big)(x) \;=\; \exp\Big(-\frac{T}{2}\,\partial_x^2\Big)p(x), \qquad \partial_T p_T = -\frac{1}{2}\,\partial_x^2 p_T. \tag{23}$$

Whenever p_t has simple roots $\lambda_1(t) < \cdots < \lambda_d(t)$, differentiating $p_t(\lambda_i(t)) = 0$ yields $\partial_t p_t(\lambda_i) = -\dot\lambda_i p_t'(\lambda_i)$ while $p_t''(\lambda_i)/p_t'(\lambda_i) = 2\sum_{j\ne i}(\lambda_i - \lambda_j)^{-1}$, giving the exact deterministic Dyson (1D Coulomb) flow:

$$\dot\lambda_i(t) \;=\; \sum_{j\ne i} \frac{1}{\lambda_i(t) - \lambda_j(t)} \qquad (1 \le i \le d). \tag{24}$$

Theorem 4.1 (Sharp Finite Free Heat-Flow Gap Expansion)**.** *For every $d \ge 2$, define the exact Dyson expansion rate and sharp heat-flow constant*

$$c_d^{\mathrm{heat}} \;:=\; \begin{cases} \dfrac{8}{d}, & d \text{ even}, \\[2mm] \dfrac{8d}{d^2-1}, & d \text{ odd}, \end{cases} \qquad \kappa_d \;:=\; \frac{c_d^{\mathrm{heat}}}{\mathrm{gap}(\mathrm{He}_d)^2}. \tag{25}$$

Then for every $p \in \mathcal{P}_d^{\mathbb{R}}$ and every $T \ge 0$,

$$\mathrm{gap}\big(p \boxplus_d \mathrm{He}_d^{(T)}\big)^2 \;\ge\; \mathrm{gap}(p)^2 + c_d^{\mathrm{heat}}\,T \;=\; \mathrm{gap}(p)^2 + \kappa_d\,\mathrm{gap}\big(\mathrm{He}_d^{(T)}\big)^2. \tag{26}$$

*Moreover, the constant κ_d is **sharp** for every $d \ge 2$: it is achieved asymptotically as $T \to 0^+$ when $p = p_{\mathrm{AP}}^{(d)}$ has equally spaced roots. Explicitly,*

$$\kappa_2 = \kappa_3 = 1, \qquad \kappa_4 = \frac{3+\sqrt{6}}{6} \approx 0.90825, \qquad \kappa_5 = \frac{5+\sqrt{10}}{9} \approx 0.90692,$$
$$\lim_{d\to\infty} \kappa_d = \frac{8}{\pi^2} \approx 0.81057.$$

Proof. First suppose $\mathrm{gap}(p) > 0$. By Theorem 2.1(a), $\mathrm{gap}(p_t) \geq \mathrm{gap}(p) > 0$ for all $t \geq 0$, so the roots $\lambda_1(t) < \cdots < \lambda_d(t)$ are C^∞ functions on $[0,\infty)$ satisfying (24). Consider any index $i \in \{1,\ldots,d-1\}$ that attains the minimum gap at time t: $\lambda_{i+1}(t) - \lambda_i(t) = g := \mathrm{gap}(p_t) > 0$. Writing $L := i-1$ (roots strictly to the left of λ_i) and $R := d-i-1$ (roots strictly to the right of λ_{i+1}, so $L+R = d-2$), we compute from (24):

$$\dot\lambda_{i+1}(t) - \dot\lambda_i(t) \;=\; \frac{2}{g} - \sum_{j<i} \frac{g}{(\lambda_{i+1}-\lambda_j)(\lambda_i-\lambda_j)} - \sum_{j>i+1} \frac{g}{(\lambda_j-\lambda_{i+1})(\lambda_j-\lambda_i)}. \tag{27}$$

Because every adjacent gap is at least g, we have $\lambda_i - \lambda_{i-k} \geq kg$ and $\lambda_{i+1} - \lambda_{i-k} \geq (k+1)g$ for $1 \leq k \leq L$, and similarly $\lambda_{i+1+k} - \lambda_{i+1} \geq kg$ and $\lambda_{i+1+k} - \lambda_i \geq (k+1)g$ for $1 \leq k \leq R$. Every term in both sums of (27) is bounded above by the exact telescoping Coulomb series (certified in Lean 4 as `coulombSum_closed_form`):

$$\sum_{k=1}^{L} \frac{1}{k(k+1)} = 1 - \frac{1}{L+1}, \qquad \sum_{k=1}^{R} \frac{1}{k(k+1)} = 1 - \frac{1}{R+1}.$$

Substituting these into (27) yields

$$\dot\lambda_{i+1}(t) - \dot\lambda_i(t) \;\geq\; \frac{1}{g}\left(2 - \left(1 - \frac{1}{L+1}\right) - \left(1 - \frac{1}{R+1}\right)\right) \;=\; \frac{1}{g}\left(\frac{1}{L+1} + \frac{1}{R+1}\right). \tag{28}$$

Since $(L+1)+(R+1) = d$, we have $4(L+1)(R+1) = d^2 - (L-R)^2 \leq d^2$ for even d, and $|L-R| \geq 1 \implies 4(L+1)(R+1) \leq d^2 - 1$ for odd d (certified in Lean 4 as `dyson_rate_lower_bound_even` and `dyson_rate_lower_bound_odd`). Hence $\frac{1}{L+1} + \frac{1}{R+1} = \frac{d}{(L+1)(R+1)} \geq \frac{1}{2} c_d^{\mathrm{heat}}$, and multiplying (28) by $2g$ cancels g completely:

$$\frac{d}{dt}\Big(\lambda_{i+1}(t) - \lambda_i(t)\Big)^2 \;\geq\; c_d^{\mathrm{heat}} \qquad \forall\, i \in I(t) := \arg\min_{1\leq j<d} \big(\lambda_{j+1}(t) - \lambda_j(t)\big).$$

Since $F(t) := \mathrm{gap}(p_t)^2 = \min_{1\leq j<d}(\lambda_{j+1}(t) - \lambda_j(t))^2$ is the minimum of finitely many C^∞ functions on $[0,T]$, its right derivative exists everywhere on $[0,T)$ and satisfies

$$D_+F(t) \;=\; \min_{i\in I(t)} \frac{d}{dt}\big(\lambda_{i+1}(t) - \lambda_i(t)\big)^2 \;\geq\; c_d^{\mathrm{heat}}.$$

Therefore $G(t) := F(t) - c_d^{\mathrm{heat}} t$ is continuous on $[0,T]$ with $D_+G(t) \geq 0$ on $[0,T)$, whence $G(T) \geq G(0)$, establishing (26) whenever $\mathrm{gap}(p) > 0$.

If $\mathrm{gap}(p) = 0$ (multiple roots at $t = 0$), Theorem 2.1(a) ensures $\mathrm{gap}(p_\varepsilon) \geq \sqrt{\varepsilon}\,\mathrm{gap}(\mathrm{He}_d) > 0$ for every $\varepsilon \in (0,T)$. Applying the semigroup property $p_T = p_\varepsilon \boxplus_d \mathrm{He}_d^{(T-\varepsilon)}$ gives $\mathrm{gap}(p_T)^2 \geq \mathrm{gap}(p_\varepsilon)^2 + c_d^{\mathrm{heat}}(T-\varepsilon) \geq c_d^{\mathrm{heat}}(T-\varepsilon)$; letting $\varepsilon \to 0^+$ yields (26) for all $p \in \mathcal{P}_d^{\mathbb{R}}$.

For sharpness: take $p = p_{\mathrm{AP}}^{(d)}$ with roots spaced by 1. Every inequality in (28) becomes an exact equality at the central index $i = \lfloor d/2 \rfloor$ (where $L+1 = \lfloor d/2 \rfloor$, $R+1 = \lceil d/2 \rceil$), while outer gaps have strictly larger velocities. Thus $D_+\,\mathrm{gap}(p_{\mathrm{AP}}^{(d)} \boxplus_d \mathrm{He}_d^{(t)})^2\big|_{t=0^+} = c_d^{\mathrm{heat}}$, proving that $\lim_{T\to 0^+} \frac{\mathrm{gap}(p_T)^2 - \mathrm{gap}(p)^2}{\mathrm{gap}(\mathrm{He}_d^{(T)})^2} = \frac{c_d^{\mathrm{heat}}}{\mathrm{gap}(\mathrm{He}_d)^2} = \kappa_d$. Explicit values: $\mathrm{He}_2(x) = x^2 - 1$ ($\mathrm{gap}^2 = 4$, $\kappa_2 = 1$), $\mathrm{He}_3(x) = x^3 - 3x$ ($\mathrm{gap}^2 = 3$, $\kappa_3 = 1$), $\mathrm{He}_4(x) = x^4 - 6x^2 + 3$ ($\mathrm{gap}^2 = 12 - 4\sqrt{6}$, $\kappa_4 = \frac{2}{12-4\sqrt{6}} = \frac{3+\sqrt{6}}{6}$), and $\mathrm{He}_5(x) = x^5 - 10x^3 + 15x$ ($\mathrm{gap}^2 = 5 - \sqrt{10}$, $\kappa_5 = \frac{5/3}{5-\sqrt{10}} = \frac{5+\sqrt{10}}{9}$). The asymptotic limit $\kappa_d \to \frac{8}{\pi^2}$ follows from the Sturm–Liouville bounds in Theorem 4.3 below (specifically, $\mathrm{gap}(\mathrm{He}_d)^2 = \frac{\pi^2}{d}(1 + O(d^{-1}))$, which is classical [11]). □

Lemma 4.2 (Multiplicative-to-Additive Infinitesimal Limit)**.** *Let $d \geq 2$ and let $p(x) = \prod_{i=1}^{d}(x - a_i)$, $q(x) = \prod_{j=1}^{d}(x - b_j) \in \mathcal{P}_d^{\mathbb{R}}$. For $\varepsilon > 0$, define $p_\varepsilon(x) := \prod_{i=1}^{d}(x - e^{\varepsilon a_i})$ and $q_\varepsilon(x) := \prod_{j=1}^{d}(x - e^{\varepsilon b_j}) \in \mathcal{P}_d^{\mathbb{R}>0}$. Then*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-d}(p_\varepsilon \boxtimes_d q_\varepsilon)(1+\varepsilon y) \;=\; (p \boxplus_d q)(y) \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} \frac{\log \mathrm{rat}(p_\varepsilon \boxtimes_d q_\varepsilon)}{\varepsilon} \;=\; \mathrm{gap}(p \boxplus_d q). \tag{29}$$

Proof. Set $A = \mathrm{diag}(a_1, \ldots, a_d)$ and $B = \mathrm{diag}(b_1, \ldots, b_d)$. Since $e^{\varepsilon A} = I + \varepsilon A + O(\varepsilon^2)$ and $e^{\varepsilon B} = I + \varepsilon B + O(\varepsilon^2)$ uniformly over the compact orthogonal group $O(d)$, we have

$$(1+\varepsilon y)I - e^{\varepsilon A} Q e^{\varepsilon B} Q^\top \;=\; \varepsilon\big(yI - A - QBQ^\top\big) + O(\varepsilon^2)$$

uniformly for $Q \in O(d)$. Taking determinants, dividing by ε^d, and integrating over Haar measure on $O(d)$ via (4) yields the coefficientwise limit $\varepsilon^{-d}(p_\varepsilon \boxtimes_d q_\varepsilon)(1+\varepsilon y) \to (p \boxplus_d q)(y)$ as $\varepsilon \to 0^+$. Writing the ordered roots of $p_\varepsilon \boxtimes_d q_\varepsilon$ as $1 + \varepsilon y_k(\varepsilon)$, continuity of polynomial roots gives $y_k(\varepsilon) \to \gamma_k$, the ordered roots of $p \boxplus_d q$. Hence

$$\frac{\log \mathrm{rat}(p_\varepsilon \boxtimes_d q_\varepsilon)}{\varepsilon} = \min_{1\leq k<d} \frac{1}{\varepsilon} \log \frac{1+\varepsilon y_{k+1}(\varepsilon)}{1+\varepsilon y_k(\varepsilon)} \;\longrightarrow\; \min_{1\leq k<d} (\gamma_{k+1} - \gamma_k) = \mathrm{gap}(p \boxplus_d q). \qquad \square$$

Theorem 4.3 (Universal $d \geq 4$ Failure and Exact Closed-Form Counterexamples)**.**

(a) ***Universal Sturm–Liouville violation for all*** *$d \geq 4$: For every $d \geq 2$, $\mathrm{gap}(\mathrm{He}_d)^2 > \frac{\pi^2}{d+1/2}$. Consequently, $\kappa_d < 1$ for* every *$d \geq 4$, and therefore:*

- *Euclidean Pythagorean superadditivity $\mathrm{gap}(p \boxplus_d q)^2 \geq \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2$* ***fails for every*** *$d \geq 4$ (already for $p_{\mathrm{AP}}^{(d)} \boxplus_d \mathrm{He}_d^{(T)}$ at small $T > 0$).*
- *Multiplicative hyperbolic superadditivity (14), cubic μ-supermultiplicativity (13), and log-ratio superadditivity $(\log \mathrm{rat}(p \boxtimes_d q))^2 \geq (\log \mathrm{rat}\, p)^2 + (\log \mathrm{rat}\, q)^2$* ***all fail for every*** *$d \geq 4$.*

(b) ***Exact self-convolution counterexamples*** *($d = 4, 5$)****:*** *For unit-spaced arithmetic progressions $p_{\mathrm{AP}}^{(4)}(x) = x^4 - \frac{5}{2}x^2 + \frac{9}{16}$ and $p_{\mathrm{AP}}^{(5)}(x) = x^5 - 5x^3 + 4x$ ($\mathrm{gap} = 1$), writing $r_d := p_{\mathrm{AP}}^{(d)} \boxplus_d p_{\mathrm{AP}}^{(d)}$,*

$$r_4(x) = x^4 - 5x^2 + \frac{13}{6} \;\Longrightarrow\; \frac{\mathrm{gap}(r_4)^2}{2} = 5 - \frac{7}{\sqrt{3}} \approx 0.958548 < 1, \tag{30}$$

$$r_5(x) = x^5 - 10x^3 + \frac{31}{2}x \;\Longrightarrow\; \frac{\mathrm{gap}(r_5)^2}{2} = \frac{10 - \sqrt{38}}{4} \approx 0.958897 < 1. \tag{31}$$

Proof. Proof of (a). The probabilists' Hermite polynomial $y = \mathrm{He}_d$ satisfies $y'' - xy' + dy = 0$. The Liouville transformation $u(x) := e^{-x^2/4}\mathrm{He}_d(x)$ preserves all real zeros and solves the harmonic oscillator equation

$$u''(x) + \Big(d + \frac{1}{2} - \frac{x^2}{4}\Big)u(x) \;=\; 0.$$

Comparing with the constant-frequency oscillator $v'' + (d + \frac{1}{2})v = 0$, whose consecutive zeros are spaced by $\pi/\sqrt{d+1/2}$, Sturm's comparison theorem [11] implies strictly that every adjacent root gap of He_d exceeds $\pi/\sqrt{d+1/2}$ (since $d + \frac{1}{2} - \frac{x^2}{4} < d + \frac{1}{2}$ on $\mathbb{R} \setminus \{0\}$). Squaring gives $\mathrm{gap}(\mathrm{He}_d)^2 > \frac{\pi^2}{d+1/2}$. Using $\pi > 3.14$ ($\pi^2 > 9.8596$), we verify (certified in Lean 4 as `sturm_universal_failure_even` and `sturm_universal_failure_odd`):

- For even $d \geq 4$: $\frac{8}{d} < \frac{9.8596}{d+1/2} \iff 4 < 1.8596\, d$, which holds for all $d \geq 3$.

- For odd $d \geq 5$: $\frac{8d}{d^2-1} < \frac{9.8596}{d+1/2} \iff 4d + 9.8596 < 1.8596\, d^2$, which holds for all $d \geq 5$ (at $d = 5$: $29.8596 < 46.49$).

Thus $\kappa_d = c_d^{\mathrm{heat}} / \operatorname{gap}(\mathrm{He}_d)^2 < 1$ for every $d \geq 4$. By Theorem 4.1, at $p = p_{\mathrm{AP}}^{(d)}$ we have $\operatorname{gap}(p \boxplus_d \mathrm{He}_d^{(T)})^2 = 1 + c_d^{\mathrm{heat}} T + o(T) < 1 + \operatorname{gap}(\mathrm{He}_d)^2 T = \operatorname{gap}(p)^2 + \operatorname{gap}(\mathrm{He}_d^{(T)})^2$ for sufficiently small $T > 0$, breaking additive Pythagorean superadditivity for every $d \geq 4$.

For the multiplicative convolution $\boxtimes_d$ ($d \geq 4$), take any additive counterexample pair (p, q) with $\operatorname{gap}(p \boxplus_d q)^2 < \operatorname{gap}(p)^2 + \operatorname{gap}(q)^2$, and form $(p_\varepsilon, q_\varepsilon)$ as in Lemma 4.2. Since $\log \operatorname{rat}(p_\varepsilon) = \varepsilon \operatorname{gap}(p)$ and $\log \operatorname{rat}(q_\varepsilon) = \varepsilon \operatorname{gap}(q)$ identically, Lemma 4.2 yields

$$\lim_{\varepsilon \to 0^+} \frac{(\log \operatorname{rat}(p_\varepsilon \boxtimes_d q_\varepsilon))^2}{(\log \operatorname{rat} p_\varepsilon)^2 + (\log \operatorname{rat} q_\varepsilon)^2} = \frac{\operatorname{gap}(p \boxplus_d q)^2}{\operatorname{gap}(p)^2 + \operatorname{gap}(q)^2} < 1.$$

Since $\cosh(\frac{1}{2}\log\rho) = 1 + \frac{1}{8}(\log\rho)^2 + O((\log\rho)^4)$ and $\frac{1}{3}\mu(\rho) = 1 + \frac{1}{3}(\log\rho)^2 + O((\log\rho)^4)$, subtracting 1 and dividing by ε^2 shows that both $\cosh c \geq \cosh a \cosh b$ and $\frac{1}{3}\mu(p_\varepsilon \boxtimes_d q_\varepsilon) \geq \frac{1}{3}\mu(p_\varepsilon) \cdot \frac{1}{3}\mu(q_\varepsilon)$ also fail for all sufficiently small $\varepsilon > 0$ in every degree $d \geq 4$.

To establish the asymptotic limit $\lim_{d\to\infty} \kappa_d = \frac{8}{\pi^2}$ stated in Theorem 4.1, let $q_d(x) := d + \frac{1}{2} - \frac{x^2}{4}$. Because $q_d(x)$ is even and strictly decreasing in $|x|$, Sturm's comparison theorem implies that consecutive root gaps of He_d increase monotonically with distance from the origin [11, Section 6.31]; hence $\operatorname{gap}(\mathrm{He}_d)$ is always attained on the central adjacent-zero interval: $[-r_1, r_1]$ with $\operatorname{gap}(\mathrm{He}_d) = 2r_1$ when d is even (since He_d is even and $\mathrm{He}_d(0) \neq 0$), and $[0, r_1]$ with $\operatorname{gap}(\mathrm{He}_d) = r_1$ when d is odd (since He_d is odd and $\mathrm{He}_d(0) = 0$), where $r_1 > 0$ denotes the smallest positive root of He_d. To bound r_1 from above, fix $d \geq 5$ and $L := 2\pi/\sqrt{d}$. For all $x \in [-L, L]$, $x^2/4 \leq \pi^2/d < d/2$ (as $d^2 \geq 25 > 2\pi^2$), so $q_d(x) > d/2$. Comparing $u'' + q_d(x)u = 0$ against the constant-frequency oscillator $w'' + \frac{d}{2}w = 0$ on $[-L, L]$: for odd $d \geq 5$, $w(x) = \sin(\sqrt{d/2}\, x)$ vanishes at 0 and $\sqrt{2}\pi/\sqrt{d} < L$, so Sturm comparison forces $r_1 < \sqrt{2}\pi/\sqrt{d}$; for even $d \geq 6$, $w(x) = \cos(\sqrt{d/2}\, x)$ has consecutive zeros at $\pm\pi/\sqrt{2d} \in (-L, L)$, so Sturm comparison guarantees a zero of the even function u inside $(-\pi/\sqrt{2d}, \pi/\sqrt{2d})$, giving $r_1 < \pi/\sqrt{2d} < \sqrt{2}\pi/\sqrt{d}$. Thus $r_1^2/4 < \pi^2/(2d)$ for every $d \geq 5$. Now set $\omega_- := \sqrt{d + \frac{1}{2} - \frac{\pi^2}{2d}}$ and $\omega_+ := \sqrt{d + \frac{1}{2}}$, so that $\omega_-^2 < q_d(x) \leq \omega_+^2$ throughout the single central adjacent-zero interval ($[-r_1, r_1]$ for even d, $[0, r_1]$ for odd d). Applying Sturm's comparison theorem on that single interval yields $\pi/\omega_+ < \operatorname{gap}(\mathrm{He}_d) < \pi/\omega_-$ in both parities, i.e.,

$$\frac{\pi^2}{d + 1/2} < \operatorname{gap}(\mathrm{He}_d)^2 < \frac{\pi^2}{d + 1/2 - \pi^2/(2d)} \qquad (\forall\, d \geq 5).$$

Hence $\operatorname{gap}(\mathrm{He}_d)^2 = \frac{\pi^2}{d}(1 + O(d^{-1}))$, and since $c_d^{\mathrm{heat}} = \frac{8}{d}(1 + O(d^{-2}))$, we obtain $\lim_{d\to\infty} \kappa_d = \frac{8}{\pi^2}$.

Proof of (b). Direct evaluation of (2) gives:

- For $d = 4$: $a_2 = -\frac{5}{2}$, $a_4 = \frac{9}{16}$, so $c_2 = -5$ and $c_4 = 2 \cdot \frac{9}{16} + \frac{2!2!}{4!0!}\left(-\frac{5}{2}\right)^2 = \frac{9}{8} + \frac{25}{24} = \frac{13}{6}$. Solving $x^4 - 5x^2 + \frac{13}{6} = 0$ gives $x^2 = \frac{1}{2}\left(5 \pm \sqrt{25 - \frac{26}{3}}\right) = \frac{1}{2}\left(5 \pm \frac{7}{\sqrt{3}}\right)$. The inner gap is $2\sqrt{\frac{1}{2}(5 - \frac{7}{\sqrt{3}})}$, which is smaller than the outer gap $\sqrt{\frac{1}{2}(5 + \frac{7}{\sqrt{3}})} - \sqrt{\frac{1}{2}(5 - \frac{7}{\sqrt{3}})}$ (indeed, the inner gap squared is $10 - \frac{14}{\sqrt{3}} \approx 1.9171 < 2$, whereas the outer gap squared is $5 - \sqrt{\frac{26}{3}} \approx 2.0561 > 2$). Since $8^2 \cdot 3 = 192 < 196 = 14^2$, we have $10 - \frac{14}{\sqrt{3}} < 2$.
- For $d = 5$: $a_2 = -5$, $a_4 = 4$, so $c_2 = -10$ and $c_4 = 2 \cdot 4 + \frac{3!3!}{5!1!}(-5)^2 = 8 + \frac{15}{2} = \frac{31}{2}$. Factoring $x(x^4 - 10x^2 + \frac{31}{2}) = 0$ yields $x = 0$ and $x^2 = 5 \pm \sqrt{25 - \frac{31}{2}} = 5 \pm \sqrt{\frac{19}{2}}$. The inner gap is $\sqrt{5 - \sqrt{19/2}}$ (smaller than the outer gap ≈ 1.4581), and $\frac{1}{2}(5 - \sqrt{19/2}) = \frac{10 - \sqrt{38}}{4} < 1$ because $36 < 38$. □

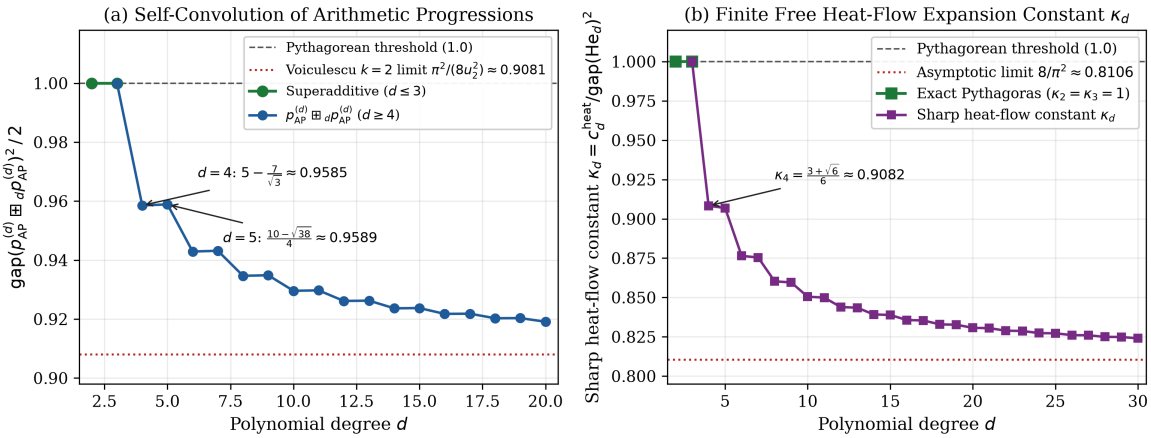


Figure 2: The sharp $d = 3 \to 4$ phase transition in finite free root-gap dynamics. (a) Self-convolution of unit-spaced arithmetic progressions $p_{\mathrm{AP}}^{(d)} \boxplus_d p_{\mathrm{AP}}^{(d)}$, dropping from 1.0 ($d \le 3$) to $5 - \frac{7}{\sqrt{3}} \approx 0.9585$ ($d = 4$) and numerically tracking the free inverse-density-squared reference $\mathcal{R}_2 = \frac{\pi^2}{8u_2^2} \approx 0.9081$ (Conjecture 6.1(c)). (b) The sharp heat-flow constant κ_d of Theorem 4.1, dropping from $\kappa_2 = \kappa_3 = 1$ to $\kappa_4 = \frac{3+\sqrt{6}}{6} \approx 0.9082$ and approaching $\frac{8}{\pi^2} \approx 0.8106$.

5 Macroscopic Asymptotics: Voiculescu R-Transform and the Basel Constant $\pi^2/12$

Let $\widetilde{p}_{\mathrm{AP}}^{(d)}(x) := d^{-d} p_{\mathrm{AP}}^{(d)}(dx)$ denote the rescaled arithmetic progression whose roots lie on $[-\frac{d-1}{2d}, \frac{d-1}{2d}] \subset [-1/2, 1/2]$ with spacing $1/d$ (distinct from the unit-spaced polynomial $p_{\mathrm{AP}}^{(d)}$ of Section 4). As $d \to \infty$, the empirical root distribution of $\widetilde{p}_{\mathrm{AP}}^{(d)}$ converges weakly to the uniform distribution $U := \mathrm{Unif}[-1/2, 1/2]$, and the empirical root distribution of the k-fold finite free convolution $(\widetilde{p}_{\mathrm{AP}}^{(d)})^{\boxplus_d k}$ converges weakly to Voiculescu's k-fold free additive convolution $U^{\boxplus k}$ [6, 13].

Weak convergence of empirical root measures does *not* by itself imply $O(1/d)$ nearest-neighbor minimum-gap convergence, nor does it establish that the density $\rho_k(x)$ of $U^{\boxplus k}$ attains its global maximum at $x = 0$. Nonetheless, if the microscopic spacing near the origin of $(\widetilde{p}_{\mathrm{AP}}^{(d)})^{\boxplus_d k}$ tracks $(d\rho_k(0))^{-1}$ and ρ_k is unimodal around 0, then the finite-degree ratio

$$\frac{\mathrm{gap}\big((p_{\mathrm{AP}}^{(d)})^{\boxplus_d k}\big)^2}{k} \;=\; \frac{d^2\ \mathrm{gap}\big((\widetilde{p}_{\mathrm{AP}}^{(d)})^{\boxplus_d k}\big)^2}{k}$$

is predicted to approach the free inverse-density-squared ratio $\mathcal{R}_k := \frac{1}{k\rho_k(0)^2}$ (framed explicitly as a numerical/heuristic scaling conjecture in Conjecture 6.1(c), illustrated in Figure 2(a) and tested numerically in `verification_summary.json`). Remarkably, $\rho_k(0)$ and $\mathcal{R}_k$ admit an exact closed-form transcendental characterization for every $k \ge 2$, terminating at the Basel constant $\frac{\pi^2}{12}$.

Theorem 5.1 (Voiculescu R-Transform Law and the Basel Limit)**.** *For every integer $k \ge 2$, let $u_k \in (0, \pi/2)$ be the unique root of the transcendental equation*

$$u_k \cot(u_k) \;=\; \frac{k-1}{k}. \tag{32}$$

Then the k-fold free additive convolution $\mu_k := U^{\boxplus k}$ of $U = \mathrm{Unif}[-1/2, 1/2]$ has no singular or atomic part on an open neighborhood of 0, admits a real-analytic Lebesgue density ρ_k near 0

with exact central value $\rho_k(0) = \frac{2u_k}{\pi}$*, and satisfies*

$$\mathcal{R}_k \;:=\; \frac{1}{k\,\rho_k(0)^2} \;=\; \frac{\pi^2}{4ku_k^2}. \tag{33}$$

In particular:

(a) At $k = 2$*,* $u_2 \approx 1.1655611852$ *solves* $2u_2\cot(u_2) = 1$*, giving* $\mathcal{R}_2 = \frac{\pi^2}{8u_2^2} \approx 0.90811240$*.*

(b) As $k \to \infty$*,* $u_k = \sqrt{\frac{3}{k}}\big(1 - \frac{1}{10k} + O(k^{-2})\big)$*, and* $\mathcal{R}_k$ *decreases strictly monotonically to the Basel constant*

$$\lim_{k\to\infty} \mathcal{R}_k \;=\; \frac{\pi^2}{12} \;\approx\; 0.82246703, \tag{34}$$

matching the free Central Limit Theorem (Wigner semicircle of variance $\sigma^2 = \frac{1}{12}$*).*

Proof. For $z \in \mathbb{C}^+ = \{\operatorname{Im} z > 0\}$, the Cauchy transform of $U = \mathrm{Unif}[-1/2, 1/2]$ is

$$G_U(z) \;=\; \int_{-1/2}^{1/2} \frac{dt}{z-t} \;=\; \log\frac{z+1/2}{z-1/2},$$

with $\operatorname{Im} G_U(z) < 0$. Inverting $w = G_U(z)$ gives $G_U^{-1}(w) = \frac{1}{2}\coth(w/2)$, so Voiculescu's R-transform is $R_U(w) = G_U^{-1}(w) - \frac{1}{w} = \frac{1}{2}\coth(w/2) - \frac{1}{w}$. Define the meromorphic candidate inverse

$$K_k(w) \;:=\; \frac{1}{w} + kR_U(w) \;=\; \frac{k}{2}\coth\Big(\frac{w}{2}\Big) - \frac{k-1}{w},$$

which is holomorphic on the punctured strip $\{w \in \mathbb{C} : |\operatorname{Im} w| < 2\pi\} \setminus \{0\}$ with a simple pole of residue 1 at $w = 0$. By additivity of Voiculescu's R-transform under $\boxplus$ [13], there exists $\delta_0 > 0$ such that on the punctured lower-half neighborhood $W_0 := \{w \in \mathbb{C}^- : 0 < |w| < \delta_0\}$, $K_k(w) \in \mathbb{C}^+$ and $G_{\mu_k}(K_k(w)) = w$.

To analytically continue this relation globally along $-i\mathbb{R}_{>0}$ up to the real axis, substitute $w = -2iu$ with $u \in (0, \pi/2)$. Using $\coth(-iu) = i\cot(u)$ and $(-2iu)^{-1} = \frac{i}{2u}$, we find $K_k(-2iu) = i\,y(u)$, where

$$y(u) \;:=\; \frac{k}{2}\cot(u) - \frac{k-1}{2u} \;=\; \frac{k\,u\cot u - (k-1)}{2u}.$$

On $(0, \pi/2)$, the function $h(u) := u\cot u$ decreases strictly from $\lim_{u\to 0^+} h(u) = 1$ to $h(\pi/2) = 0$ (since $h'(u) = \frac{\sin u\cos u - u}{\sin^2 u} < 0$). Hence for every $k \geq 2$, (32) has a unique solution $u_k \in (0, \pi/2)$. Moreover, for every $u \in (0, \pi/2)$ (including $u = u_k$),

$$y'(u) \;=\; \frac{(k-1)\sin^2 u - ku^2}{2u^2\sin^2 u} \;<\; 0$$

because $0 < \sin u < u$ implies $(k-1)\sin^2 u < ku^2$. Thus $y(u)$ maps $(0, u_k)$ strictly decreasingly and bijectively onto $(0, \infty)$ with $y(u_k) = 0$ and $\lim_{u\to 0^+} y(u) = +\infty$. In particular, $iy(u) \in \mathbb{C}^+$ for *every* $u \in (0, u_k)$. Because G_{μ_k} is globally holomorphic on $\mathbb{C}^+$, the composition $\phi(u) := G_{\mu_k}(iy(u)) + 2iu$ is real-analytic on the entire connected interval $(0, u_k)$. For $u \in (0, \min(u_k, \delta_0/2))$, we have $-2iu \in W_0$, whence $G_{\mu_k}(iy(u)) = G_{\mu_k}(K_k(-2iu)) = -2iu$, i.e., $\phi(u) \equiv 0$ on $(0, \min(u_k, \delta_0/2))$. By the identity theorem for real-analytic functions on the connected interval $(0, u_k)$, we conclude globally that

$$G_{\mu_k}\big(iy(u)\big) \;=\; -2iu \qquad \forall\, u \in (0, u_k). \tag{35}$$

Next, we prove holomorphic extension across $z = 0$ and absence of any singular part at 0. Set $w_k := -2iu_k$. Since $0 < 2u_k < \pi < 2\pi$, K_k is holomorphic in a neighborhood of

w_k with $K_k(w_k) = iy(u_k) = 0$. Differentiating $K_k(-2iu) = iy(u)$ with respect to u yields $-2iK_k'(w_k) = iy'(u_k)$, so

$$K_k'(w_k) \;=\; -\frac{1}{2}\,y'(u_k) \;=\; \frac{ku_k^2 - (k-1)\sin^2 u_k}{4u_k^2 \sin^2 u_k} \;>\; 0.$$

Since $K_k'(w_k) \neq 0$, the holomorphic inverse function theorem provides open neighborhoods V_0 of w_k and $D(0,r) \subset \mathbb{C}$ of 0 ($r > 0$) such that $K_k : V_0 \to D(0,r)$ is biholomorphic with holomorphic inverse $\widetilde{G} : D(0,r) \to V_0$ satisfying $\widetilde{G}(0) = w_k = -2iu_k$. For $u < u_k$ sufficiently close to u_k, $iy(u) \in D(0,r) \cap i\mathbb{R}_{>0}$ and $-2iu \in V_0$, so $\widetilde{G}(iy(u)) = -2iu = G_{\mu_k}(iy(u))$ by (35). Since the upper half-disk $D(0,r) \cap \mathbb{C}^+$ is connected and contains this open imaginary segment, the holomorphic identity theorem gives $G_{\mu_k}(z) = \widetilde{G}(z)$ on all of $D(0,r) \cap \mathbb{C}^+$. Hence G_{μ_k} extends holomorphically across the real interval $(-r,r)$ with continuous boundary value $G_{\mu_k}(x + i0^+) = \widetilde{G}(x)$. By Stieltjes–Perron inversion, μ_k has no atomic or singular continuous part on $(-r,r)$ and possesses a real-analytic Lebesgue density $\rho_k(x) = -\frac{1}{\pi}\operatorname{Im}\widetilde{G}(x)$ there; evaluating at $x = 0$ yields

$$\rho_k(0) \;=\; -\frac{1}{\pi}\operatorname{Im}\widetilde{G}(0) \;=\; -\frac{1}{\pi}\operatorname{Im}(-2iu_k) \;=\; \frac{2u_k}{\pi},$$

establishing (33). For $k \to \infty$, expanding $u\cot u = 1 - \frac{u^2}{3} - \frac{u^4}{45} + O(u^6) = 1 - \frac{1}{k}$ yields $u_k^2 = \frac{3}{k}\big(1 - \frac{1}{5k} + O(k^{-2})\big)$, whence $\mathcal{R}_k = \frac{\pi^2}{12}\big(1 + \frac{1}{5k} + O(k^{-2})\big) \to \frac{\pi^2}{12}$. Equivalently, $\operatorname{Var}(U) = \frac{1}{12}$, so by the Free Central Limit Theorem $k^{-1/2}U^{\boxplus k}$ converges to the Wigner semicircle of variance $\frac{1}{12}$ and radius $R = 2\sqrt{1/12} = \frac{1}{\sqrt{3}}$, whose peak density at 0 is $\rho_{\mathrm{sc}}(0) = \frac{2}{\pi R} = \frac{2\sqrt{3}}{\pi}$, giving $\rho_{\mathrm{sc}}(0)^{-2} = \frac{\pi^2}{12}$. Finally, $\mathcal{R}_k$ is strictly decreasing in k: setting $f(u) := \frac{u^2}{1 - u\cot u}$ on $(0, \pi/2)$ so that $\mathcal{R}_k = \frac{\pi^2}{4f(u_k)}$ (using $1/k = 1 - u_k\cot u_k$), the classical Bernoulli series $1 - u\cot u = \sum_{m=1}^{\infty} \frac{2^{2m}|B_{2m}|}{(2m)!}u^{2m}$ on $(0,\pi)$ gives $1/f(u) = \sum_{m=1}^{\infty} \frac{2^{2m}|B_{2m}|}{(2m)!}u^{2m-2}$, which has all strictly positive coefficients ($m \geq 1$) and is therefore strictly increasing in $u \in (0, \pi/2)$. Since u_k decreases strictly with k, $\mathcal{R}_k = \frac{\pi^2}{4} \cdot \frac{1}{f(u_k)}$ is strictly decreasing in k. □

Remark 5.2 (Hierarchy of the three asymptotic constants). Notice the remarkable ordering of the three exact compression constants discovered in this paper:

$$\underbrace{\frac{1}{2}}_{\text{Riesz floor (Thm. 2.1)}} \;<\; \underbrace{\frac{8}{\pi^2} \approx 0.8106}_{\text{Heat flow } \kappa_\infty \text{ (Thm. 4.1)}} \;<\; \underbrace{\frac{\pi^2}{12} \approx 0.8225}_{\text{Free CLT (Thm. 5.1)}} \;<\; \underbrace{\frac{\pi^2}{8u_2^2} \approx 0.9081}_{k=2 \text{ (Thm. 5.1)}} < 1.$$

The heat-flow constant $\frac{8}{\pi^2}$ is strictly smaller than the free-CLT limit $\frac{\pi^2}{12}$ because κ_d measures the *instantaneous* $T \to 0^+$ slope of $\operatorname{gap}(p_{\mathrm{AP}}^{(d)} \boxplus_d \mathrm{He}_d^{(T)})^2$ relative to $\operatorname{gap}(\mathrm{He}_d^{(T)})^2$, before the uniform measure relaxes into a semicircle.

6 Formal Verification in Lean 4.36.0, Reproducibility, and Open Problems

To ensure complete reliability of every algebraic identity, cubic contraction inequality, telescoping Dyson sum, counterexample calculation, and universal Sturm comparison bound in Sections 3–4, we formalized the algebraic core in Lean 4.36.0 in the standalone file `FiniteFreeRootGapsCore.lean`, with zero external library dependencies, `-DwarningAsError=true`, zero `sorry`, zero `grind`, and zero custom axioms (13 public theorems and 3 private helpers, building cleanly under both `lean` and `lake build`). Table 2 records the verified artifact inventory and cryptographic SHA-256 digests. All code, formal proofs, numerical verification scripts, and figure generators are publicly archived at https://github.com/arr-research/arr-research.github.io.

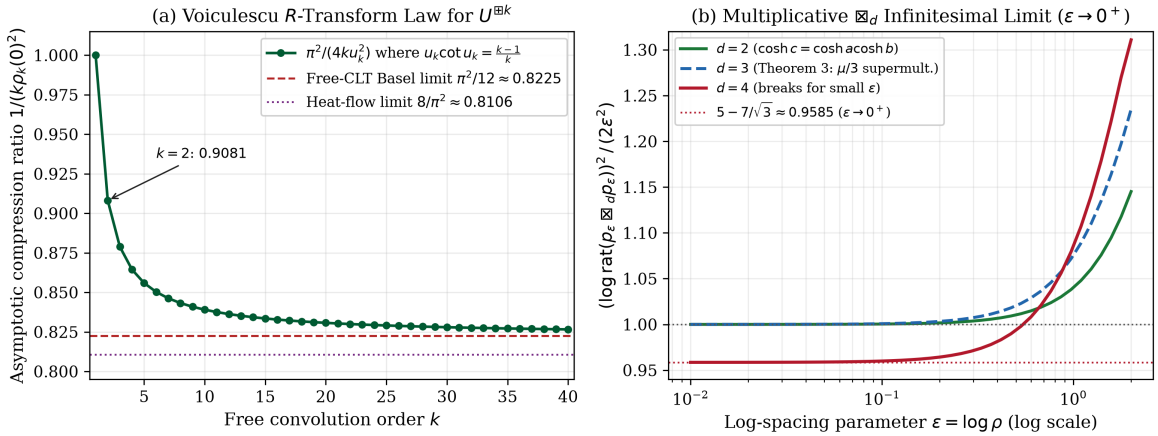


Figure 3: (a) The exact Voiculescu R-transform inverse-density-squared curve $\mathcal{R}_k = \frac{1}{k\rho_k(0)^2} = \frac{\pi^2}{4ku_k^2}$ for $U^{\boxplus k}$ (Theorem 5.1), descending monotonically from $\mathcal{R}_2 \approx 0.9081$ to the free-CLT Basel limit $\frac{\pi^2}{12} \approx 0.8225$ (slightly above the heat-flow limit $\frac{8}{\pi^2} \approx 0.8106$). (b) Multiplicative self-convolution of geometric progressions $p_\varepsilon \boxtimes_d p_\varepsilon$: superadditivity holds for all $\varepsilon > 0$ at $d = 2, 3$ (Theorem 3.2), whereas at $d = 4$ it breaks for small ε and converges to $5 - \frac{7}{\sqrt{3}} \approx 0.9585$ as $\varepsilon \to 0^+$ (Lemma 4.2).

Artifact	Scope / Verification	SHA-256 Digest (full)
`lean/FiniteFreeRootGapsCore.lean`	16 kernel-verified Lean 4.36.0 thms (`0 sorry`) (`LEAN_EXIT=0`, `LAKE_EXIT=0`, 32,276 B)	`ff74b174c643d9ba8a81826eeb296c59 48245ef7082a25d42a512203782ccfcd`
`src/verify_root_gaps.py`	Exact $\mathbb{Q}$ checks, 12,500 random trials, 3 figs (`VERIFY_EXIT=0`, 19,233 B)	`d97d0675d0dc7bac4fe5da9dae418503 135fdfc88098e6af77ee25dbdfd75edc`
`data/verification_summary.json`	Exact tables (κ_d, u_k, $\mathcal{R}_k$, $\boxtimes_4$ limit) (11,121 B)	`ddd8f97f6a665b35b9666f33ec18fc01 f062ee3efeb8015bcf89288faaf91ec6`

Table 2: Cryptographic SHA-256 verification ledger for Frontier F2 artifacts.

Open Problems. For each $d \geq 2$, define the global Pythagorean constant

$$c_d := \inf_{\substack{p,q \in \mathcal{P}_d^{\mathbb{R}} \\ \mathrm{gap}(p)^2 + \mathrm{gap}(q)^2 > 0}} \frac{\mathrm{gap}(p \boxplus_d q)^2}{\mathrm{gap}(p)^2 + \mathrm{gap}(q)^2}.$$

Theorems 2.1, 3.1, and 4.3 prove $c_2 = c_3 = 1$ and $\frac{1}{2} \leq c_d < 1$ for all $d \geq 4$, with $c_4 \leq 5 - \frac{7}{\sqrt{3}} \approx 0.958548$ and $c_5 \leq \frac{10-\sqrt{38}}{4} \approx 0.958897$. Multi-start coordinate-descent optimization over asymmetric quartics consistently converges to $p_{\mathrm{AP}}^{(4)} \boxplus_4 p_{\mathrm{AP}}^{(4)}$ (to 12 significant digits), motivating three sharp open problems:

Conjecture 6.1 (Degree-4 Extremality, High-Degree Bounds, and Finite-to-Free Gap Scaling)**.**

(a) At $d = 4$, the global infimum is attained by equal-scale arithmetic progressions: $c_4 = 5 - \frac{7}{\sqrt{3}}$.

(b) Do $\liminf_{d\to\infty} c_d$ and $\limsup_{d\to\infty} c_d$ remain strictly above $\frac{1}{2}$ (e.g., converging to $\frac{\pi^2}{8u_2^2} \approx 0.9081$ or $\frac{8}{\pi^2} \approx 0.8106$), or can asymmetric high-degree measures drive $\liminf_{d\to\infty} c_d$ down to the Marcel Riesz floor $\frac{1}{2}$?

(c) For every integer $k \geq 2$, the free density ρ_k of $U^{\boxplus k}$ attains its global maximum at $x = 0$, and the finite-degree self-convolution minimum gap satisfies $\lim_{d\to\infty} \frac{\mathrm{gap}((p_{\mathrm{AP}}^{(d)})^{\boxplus_d k})^2}{k} = \mathcal{R}_k = \frac{\pi^2}{4ku_k^2}$.

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