Tropical Spanning Criteria, Exact $\mathbb{F}_2$-Polynomial No-Strings Certificates, and Machine-Checked Energy-Barrier Bounds for 3D Translation-Invariant Stabilizer Codes

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Abstract

Certifying that a three-dimensional translation-invariant CSS or trivariate bicycle (TB) stabilizer code is a genuine Type-II fracton phase—free of mobile string-like logical operators along *every* lattice direction $v \in \mathbb{Z}^3 \setminus \{0\}$ and of *every* period $p \geq 1$—is the algebraic gatekeeper for logarithmic energy barriers and partial self-correction. Recent universal certification frameworks (e.g., Xu & Xu, 2026) and formal QEC verification efforts highlight a persistent gap: directional tests restricted to coordinate axes or short lattice vectors can be fooled by higher-slope "stealth strings," while floating-point or heuristic solvers leave formal soundness gaps. We close this gap with three tightly integrated contributions. First, we prove the **Tropical Spanning Criterion** (Theorem 2.3): because every element of $\overline{\mathbb{F}}_2^{\times}$ is a root of unity, a nontrivial coprime 2-generator 3D translation-invariant CSS code $(f, g) \in \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]^2$ ($\gcd(f, g) = 1$, $\langle f, g \rangle \neq R$) satisfies the Universal All-Direction No-Strings Condition *if and only if* every geometrically irreducible curve component $C_i \subset V(f, g) \subset (\overline{\mathbb{F}}_2^{\times})^3$ has a full-rank tropical valuation fan, $\mathrm{rank}_{\mathbb{Q}}(\mathrm{Trop}(C_i)) = 3$. Second, we introduce a zero-floating-point-trust certificate format (`FRACTON-CERT-1.0`) combining exact $\mathbb{F}_2[x, y, z]$ resultant Bézout cofactors, six $\mathbb{F}_2[t]$ axis-content Bézout witnesses, exact $\mathbb{F}_4(t)$ birational parameterizations with explicit rational inverses $t = N(x, y)/D(x, y)$ (Proposition 3.1, blocking reducible-factor forgeries by reduced-degree matching), exact component completeness $\prod_i H_i(x, y) = R_z(x, y)$, and exact place-valuation ray matrices with $\det(W) \neq 0$, alongside dual coprime $\mathbb{F}_2[x, y, z]/\mathbb{F}_4$-point obstruction certificates for string-harboring codes. Third, we mechanize the algebraic, tropical, and scale-ladder energy-barrier core—16 theorems culminating in the conditional logarithmic bound $B(d) \geq c_1 \lfloor \log_{2\beta+3}(d/R_0) \rfloor$ (Theorem 4.1)—in zero-dependency Lean 4 (`v4.35.0`) with **0 `sorry`** and **0 `Classical.choice`**. Across a systematic scan of 155 weight-4 and weight-5 3D TB codes (strictly separating 1 non-coprime code and 10 trivial product-state codes with $\langle f, g \rangle = R$, leaving 144 nontrivial codes), **24/103 (23.3%)** of nontrivial codes passing all coordinate-axis checks—and **23/102 (22.5%)** passing all 13 unit-box directions ($\|v\|_\infty = 1$)—harbor $\|v\|_\infty = 2$ stealth rigid strings such as $v = (0, -1, 2)$ and $v = (-2, -1, 2)$, collapsing their energy barriers to $E_{\mathrm{string}} \leq 6$ despite fracton-like finite-torus homology $k(L)$; extending the screen to $\|v\|_\infty = 3$ exposes 12 additional traps, raising nontrivial false-positive rates to **35.0%** (axis-only) and **34.3%** (unit-box). All 8 published atlas certificates verify with zero floating-point operations and reject 26/26 live adversarial forgery and type-injection attacks.

1 Introduction

Self-correcting quantum memories aim to protect encoded quantum information passively against thermal noise via macroscopic energy barriers between orthogonal ground states [5, 14, 3, 4]. In

*Independent researcher (Sweden). Prepared with AI assistance (Google Antigravity). Code, Lean 4 proofs, and exact $\mathbb{F}_2/\mathbb{F}_4/\mathbb{Z}$ certificates: https://github.com/lluiseriksson/fracton-barrier-certificates (MIT license).

three spatial dimensions, Haah's Cubic Code A [7] established the paradigm of *Type-II fracton* stabilizer codes: 3D translation-invariant CSS Hamiltonians with *no string-like logical operators*, forcing any sequence of single-qubit Pauli operations mapping between logical states on a distance-d lattice to overcome a logarithmically growing energy barrier $E_{\min}(d) = \Omega(\log d)$ [3, 11, 12]. In Haah's Laurent-polynomial module formalism [8], a 2-generator CSS / trivariate bicycle (TB) code over $R = \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ is specified by two coprime polynomials $(f, g) \in R^2$ with $\langle f, g\rangle \neq R$, and the absence of mobile excitations is equivalent to the algebraic *no-strings condition*: for every nonzero lattice vector $v = (v_x, v_y, v_z) \in \mathbb{Z}^3 \setminus \{0\}$ and every period $p \geq 1$, the shifted binomial $1 + u_v^p$ (where $u_v = x^{v_x} y^{v_y} z^{v_z}$) is a non-zero-divisor in the excitation module $R/\langle f, g\rangle$.

Recent breakthroughs in November 2026 have sharpened both the physics and formal-verification frontiers of this problem. Xu & Xu [13] proposed universal certification criteria for partially self-correcting 3D translation-invariant stabilizer codes, while high-rate bivariate and trivariate bicycle LDPC architectures [2] and formal QEC libraries in Lean 4 [9] have underscored the demand for machine-checkable guarantees rather than heuristic numerical screening. Yet certifying the *universal* quantifier over *all* directions $v \in \mathbb{Z}^3 \setminus \{0\}$ and *all* periods $p \geq 1$ faces three sharp mathematical and engineering obstacles:

1. **Stealth Strings Beyond Coordinate Axes and Unit-Box Diagonals.** Testing only the three coordinate axes $(1,0,0), (0,1,0), (0,0,1)$—or even all 13 primitive directions in the unit box $\|v\|_\infty = 1$—cannot rule out higher-slope rigid strings such as $y + z^2 \in \langle f, g\rangle$ $(v = (0, -1, 2))$ or $x^2 y + z^2 \in \langle f, g\rangle$ $(v = (-2, -1, 2))$. Worse, stealth-string codes can exhibit wildly oscillating finite-torus logical qubit counts $k(L)$ indistinguishable by eye from Type-II fractons, while naive scans that omit torus-unit filtering can also misclassify trivial product-state codes $(\langle f, g\rangle = R)$ as fractons.
2. **Component Completeness and Reducible-Factor Forgery.** Checking a single algebraic branch of the characteristic variety $V(f, g) \subset (\overline{\mathbb{F}}_2^\times)^3$ is unsound if an omitted or reducible factor hides a mobile excitation sector. Any independently checkable certificate must certify both that *every* irreducible component has been accounted for and that no claimed component is secretly reducible.
3. **Floating-Point and Informal Proof Gaps.** Numerical Gröbner or tropical approximations over $\mathbb{C}$ cannot be trusted in characteristic 2, and hand-waved energy-barrier derivations leave subtle gaps between algebraic no-strings criteria and hierarchical cluster-merging inductions.

Contributions. We resolve all three obstacles within a single exact, machine-checked framework:

- **Exact Tropical Spanning Criterion (Theorem 2.3).** Leveraging the defining arithmetic property of $\overline{\mathbb{F}}_2$—every nonzero scalar $c \in \overline{\mathbb{F}}_2^\times$ is a root of unity—we prove that the Universal All-Direction No-Strings Condition holds for nontrivial coprime $(f, g) \in R^2$ *if and only if* every geometrically irreducible component $C_i \subset V(f, g) \subset (\overline{\mathbb{F}}_2^\times)^3$ has $\operatorname{rank}_{\mathbb{Q}}(\operatorname{Trop}(C_i)) = 3$. Conversely, any rank-$\leq 2$ component produces an exact primitive kernel vector $v \in \operatorname{Trop}(C_i)^\perp \cap (\mathbb{Z}^3 \setminus \{0\})$ exposing a rigid string operator.
- **Zero-Float `FRACTON-CERT-1.0` Certificates & Birational Irreducibility (Section 3).** We design a self-contained exact $\mathbb{F}_2/\mathbb{F}_4/\mathbb{Z}$ certificate format and pure-Python verifier with *zero floating-point operations* and recursive rejection of any `float` or `bool` anywhere in the JSON tree. Using exact birational inverses $t = N(x, y)/D(x, y)$ and reduced $\mathbb{F}_4[t]$ GCD degree matching (Proposition 3.1), the verifier proves geometric irreducibility of each plane-curve factor $H_i(x, y)$ and exact completeness $\prod_i H_i(x, y) = R_z(x, y)$, closing reducible-component forgery attacks completely. Dual coprime obstruction certificates prove string existence via exact $\mathbb{F}_2[x, y, z]$ identities $p_c f + q_c g = (m_1 + m_2)S$ paired with an $\mathbb{F}_4$-torus point $P \in V(f, g)$ with $S(P) \neq 0$.

- **Zero-Dependency Lean 4 Kernel (Section 4).** We formalize 16 core theorems in `lean/FractonBarrier.lean` on `leanprover/lean4:v4.35.0` [6] with **0** `sorry`, **0** `Classical.choice`, and zero external library dependencies, covering CSS commutation, Bézout-scaled Koszul elimination, homomorphism soundness on $\langle f, g\rangle$, $\mathbb{F}_4$-point nontriviality, axis-content refutation, exact 3×3 integer Cramer tropical spanning, tropical kernel orthogonality, conditional hierarchical logarithmic barrier induction, and finite-torus homology.
- **8-Code Certified Atlas & 155-Code 3D TB Stealth-String Scan (Section 5).** We certify four Type-II fracton codes (Haah Cubic Code A, `TB-Fracton-5`, `TB-Fracton-7`, `TB-Fracton-11`) and four string-harboring codes (including two stealth traps passing all 13 unit-box checks), compute exact $\mathbb{F}_2$ torus homology $k(L)$ for $L = 2 \ldots 9$, and quantify a **23.3%** (through $\|v\|_\infty \le 2$) to **35.0%** (through $\|v\|_\infty \le 3$) axis-only false-positive rate and **22.5%** to **34.3%** unit-box stealth-string false-positive rate among the 144 nontrivial 3D TB codes.

2 Algebraic Framework and the Tropical Spanning Criterion

2.1 Laurent-Polynomial Module Formalism for 2-Generator 3D CSS Codes

Let $R = \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be the Laurent polynomial ring over $\mathbb{F}_2$ representing translations on the 3D cubic lattice $\mathbb{Z}^3$, with involution $\overline{h}(x, y, z) = h(x^{-1}, y^{-1}, z^{-1})$. A 2-generator translation-invariant CSS (equivalently, 3D trivariate bicycle) code places two physical qubits per unit cell and assigns one X-type and one Z-type stabilizer check per site via parity-check maps $\sigma_X : R^2 \to R$ and $\sigma_Z : R^2 \to R$ defined by

$$\sigma_X(a, b) = \overline{f}\, a + \overline{g}\, b, \qquad \sigma_Z(c, d) = g\, c + f\, d, \tag{1}$$

for polynomials $f, g \in \mathbb{F}_2[x, y, z] \subset R$ (shifted to nonnegative exponents without loss of generality). Exact CSS commutation holds unconditionally in characteristic 2 because the symplectic inner product between the X-check row (f, g) and the Z-check row $(\overline{g}, \overline{f})$ is

$$f\,\overline{\overline{g}} + g\,\overline{\overline{f}} = fg + gf = 0 \quad \text{in } R \tag{2}$$

(mechanized in Lean as `css_exact_commutation`). A Z-type Pauli error supported on sublattices $(p, q) \in R^2$ produces the X-check syndrome $\epsilon(p, q) = pf + qg \in R$ (in `lean/FractonBarrier.lean` we equivalently write `syndrome f g a b = g * a + f * b` with stabilizer (fq, gq), obtained by swapping the two sublattice entries $(p, q) \leftrightarrow (b, a)$). Local Z-stabilizers (gh, fh) satisfy $\epsilon(gh, fh) = (gf + fg)h = 0$ (`stabilizer_in_kernel`). Given any Bézout identity $uf + vg = r$, every kernel element satisfies the Bézout-scaled Koszul elimination identity $ra = f(ua + vb)$ and $rb = g(ua + vb)$ (mechanized as `koszul_local_exactness`); when $\gcd(f, g) = 1$ in the UFD R, cancellation of r yields exactness of the Koszul complex at R^2, so every finite-support zero-syndrome operator is a local stabilizer and the topological charge module is $M = R/\langle f, g\rangle$.

Definition 2.1 (Universal All-Direction No-Strings Condition)**.** For $v = (v_x, v_y, v_z) \in \mathbb{Z}^3 \setminus \{0\}$, write $u_v = x^{v_x} y^{v_y} z^{v_z} \in R^\times$. A nontrivial coprime 2-generator CSS code (f, g) ($\gcd(f, g) = 1$, $\langle f, g\rangle \neq R$) admits a *rigid string operator* of direction v and period $p \ge 1$ if there exists a nontrivial excitation $S \in R \setminus \langle f, g\rangle$ such that

$$(1 + u_v^p)\, S \in \langle f, g\rangle, \tag{3}$$

i.e., there exist cofactors $p_c, q_c \in R$ with $p_c f + q_c g = (1 + u_v^p)S$ and $S \notin \langle f, g\rangle$. The code satisfies the *Universal All-Direction No-Strings Condition* if no such (v, p, S) exists for any $v \in \mathbb{Z}^3 \setminus \{0\}$ and any $p \ge 1$.

Lemma 2.2 (Bézout Membership and $\mathbb{F}_4$-Point Nontriviality)**.** *Let $f, g, S \in \mathbb{F}_2[x, y, z]$.*

(a) If $p_c f + q_c g = (m_1 + m_2)S$ for distinct monomials $m_1, m_2 \in \mathbb{F}_2[x, y, z]$ with exponent difference $v = \exp(m_2) - \exp(m_1) \neq 0$, then $(1 + u_v)S \in \langle f, g \rangle$ in R.

(b) If there exists a point $P = (p_x, p_y, p_z) \in (\overline{\mathbb{F}}_2^{\times})^3$ (in particular, $P \in (\mathbb{F}_4^{\times})^3$) such that $f(P) = 0$, $g(P) = 0$, and $S(P) \neq 0$, then $S \notin \langle f, g \rangle$ in R.

Proof. Part (a) follows by multiplying by $m_1^{-1} \in R^{\times}$. For (b), any $h = af + bg \in \langle f, g \rangle \subset R$ evaluates to $h(P) = a(P)f(P) + b(P)g(P) = 0$ at $P \in (\overline{\mathbb{F}}_2^{\times})^3$ (mechanized as `bezout_ideal_membership`), contradicting $S(P) \neq 0$ (mechanized in Lean as `point_eval_certifies_nontrivial_syndrome`). □

2.2 Why Coordinate-Axis Checks Are Necessary but Not Sufficient

Before stating the universal theorem, we note the exact algebraic certificate for coordinate-axis strings. Eliminating each variable in turn via exact Bézout cofactors $u_z f + v_z g = R_z(x, y) \neq 0$, $u_y f + v_y g = R_y(x, z) \neq 0$, and $u_x f + v_x g = R_x(y, z) \neq 0$ unconditionally certifies $\gcd(f, g) = 1$ in $\mathbb{F}_2[x, y, z]$ (any non-constant common factor $h \mid f, g$ depends on at least one variable x_k and must divide R_{x_k}, contradicting $R_{x_k} \neq 0$). Expanding $R_z(x, y) = \sum_{m=0}^{M} c_m(x)\, y^m$, if there exist $a_0(x), \ldots, a_M(x) \in \mathbb{F}_2[x]$ such that

$$\sum_{m=0}^{M} a_m(x)\, c_m(x) = x^e \quad \text{for some } e \geq 0, \tag{4}$$

then no nonzero $x_0 \in \overline{\mathbb{F}}_2^{\times}$ can annihilate all $c_m(x)$ simultaneously (Lean theorem `axis_content_bezout_refutes_constant_x`). Hence $V(f, g)$ contains no curve component on which x is constant, ruling out all x-axis strings $(1 + x^p)S \in \langle f, g \rangle$ for every $p \geq 1$ simultaneously. Applying (4) to all six bivariate resultants rules out all coordinate-axis strings—yet leaves oblique directions v completely unguarded.

2.3 The Tropical Spanning Criterion

Let $V(f, g) = \{P \in (\overline{\mathbb{F}}_2^{\times})^3 : f(P) = g(P) = 0\}$ be the characteristic variety in the 3D algebraic torus over $\overline{\mathbb{F}}_2$. When $\gcd(f, g) = 1$ and $\langle f, g \rangle \neq R$ (so $V(f, g) \neq \emptyset$ by Hilbert's Nullstellensatz), Krull's Principal Ideal Theorem implies that every irreducible component C_i ($i = 1, \ldots, r$, $r \geq 1$) of $V(f, g)$ is an algebraic curve ($\dim C_i = 1$). Let $\widetilde{C}_i$ be the smooth projective normalization of C_i, and for each discrete valuation (place) ν of the function field $\overline{\mathbb{F}}_2(C_i)$ centered at a puncture of $C_i \subset (\overline{\mathbb{F}}_2^{\times})^3$ (where at least one coordinate has a zero or pole), define the integer *tropical valuation ray*

$$w_\nu = (\nu(x),\, \nu(y),\, \nu(z)) \in \mathbb{Z}^3 \setminus \{0\}. \tag{5}$$

By the Bieri–Groves theorem [1, 10], the tropicalization $\mathrm{Trop}(C_i) \subset \mathbb{R}^3$ is the union of the rays $\mathbb{R}_{\geq 0}\, w_\nu$ over all punctures ν of $\widetilde{C}_i$. We now establish our central characterization theorem.

Theorem 2.3 (Tropical Spanning Criterion for 3D Stabilizer Codes over $\mathbb{F}_2$)**.** *Let $f, g \in R = \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be coprime ($\gcd(f, g) = 1$) with $\langle f, g \rangle \neq R$, so that $V(f, g) \subset (\overline{\mathbb{F}}_2^{\times})^3$ is a nonempty union of $r \geq 1$ geometrically irreducible curves $C_1, \ldots, C_r$. Then (f, g) satisfies the Universal All-Direction No-Strings Condition (Definition 2.1)* ***if and only if*** *for every $i \in \{1, \ldots, r\}$,*

$$\mathrm{rank}_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}\{w_\nu : \nu \in \textit{punctures}(\widetilde{C}_i)\}) = 3, \tag{6}$$

equivalently, each component C_i possesses three valuation rays $w_1, w_2, w_3 \in \mathbb{Z}^3$ satisfying $\det[w_1, w_2, w_3] \neq 0$.

Proof. In the Noetherian UFD R, $\gcd(f,g)=1$ and $\langle f,g\rangle \neq R$ imply that $\langle f,g\rangle$ is unmixed of height 2 (every associated prime $\mathfrak{p} \in \mathrm{Ass}_R(R/\langle f,g\rangle)$ is a minimal prime over $\langle f,g\rangle$ of height 2; see [8]). An element $1+u_v^p$ is a zero-divisor in $R/\langle f,g\rangle$ if and only if $1+u_v^p \in \mathfrak{p}$ for some minimal prime $\mathfrak{p}$ over $\langle f,g\rangle$, which holds over $\overline{\mathbb{F}}_2$ if and only if $1+u_v^p$ vanishes identically on some geometrically irreducible curve component $C_i \subset V(f,g)$—that is, the monomial function $u_v = x^{v_x}y^{v_y}z^{v_z}$ is a *constant* $c \in \overline{\mathbb{F}}_2^{\times}$ with $c^p = 1$ on C_i.

(⇐) Sufficiency. Suppose every C_i admits three rays $w_1, w_2, w_3 \in \mathbb{Z}^3$ with $\det[w_1,w_2,w_3] \neq 0$, and suppose for contradiction that $u_v = c \in \overline{\mathbb{F}}_2^{\times}$ is constant on C_i for some $v \in \mathbb{Z}^3 \setminus \{0\}$. Since every discrete valuation ν vanishes on nonzero constants ($\nu(c)=0$), we obtain

$$v \cdot w_j = v_x\nu_j(x) + v_y\nu_j(y) + v_z\nu_j(z) = \nu_j(u_v) = \nu_j(c) = 0 \qquad (j=1,2,3). \tag{7}$$

By the exact 3×3 integer Cramer identity (mechanized in Lean as `tropical_spanning_refutes_all_directions`), $\det[w_1,w_2,w_3] \neq 0$ forces $v=(0,0,0)$, a contradiction. Hence no $(1+u_v^p)$ can vanish on any C_i, certifying the Universal All-Direction No-Strings Condition for every $v \in \mathbb{Z}^3\setminus\{0\}$ and every $p \geq 1$.

(⇒) Necessity (Finite-Field Key). Suppose some component C_i satisfies $\mathrm{rank}_{\mathbb{Q}}(\mathrm{Trop}(C_i)) \leq 2$. Because all valuation rays w_ν lie in $\mathbb{Z}^3$, the orthogonal complement $\mathrm{Trop}(C_i)^{\perp} \subset \mathbb{Q}^3$ has dimension ≥ 1 and therefore contains a nonzero primitive integer vector $v=(v_x,v_y,v_z) \in \mathbb{Z}^3\setminus\{0\}$ satisfying $v\cdot w_\nu = 0$ for *every* puncture ν of $\widetilde{C}_i$ (in particular, for any two spanning rays w_1, w_2, the cross product $v = w_1 \times w_2$ satisfies $v\cdot w_1 = v\cdot w_2 = 0$, mechanized as `tropical_kernel_witness_orth`). Consequently, the rational function $u_v = x^{v_x}y^{v_y}z^{v_z} \in \overline{\mathbb{F}}_2(C_i)^{\times}$ has zero order of vanishing and zero pole order at every puncture of $\widetilde{C}_i$, and has neither zeros nor poles on $C_i \subset (\overline{\mathbb{F}}_2^{\times})^3$ since x,y,z are invertible on the torus. Thus u_v is a regular function on the projective curve $\widetilde{C}_i$, hence a nonzero constant $c \in \overline{\mathbb{F}}_2^{\times}$. Crucially, because $\overline{\mathbb{F}}_2$ is the algebraic closure of the *finite field* $\mathbb{F}_2$, every $c \in \overline{\mathbb{F}}_2^{\times}$ lies in some finite subfield $\mathbb{F}_{2^m}^{\times}$ and therefore satisfies $c^{2^m-1}=1$. Setting $p = 2^m-1 \geq 1$, the binomial $1+u_v^p$ vanishes identically on C_i, hence $1+u_v^p$ belongs to the corresponding minimal prime $\mathfrak{p}_i$ over $\langle f,g\rangle$, making $1+u_v^p$ a zero-divisor in $R/\langle f,g\rangle$ and yielding a rigid string operator of direction v and period dividing 2^m-1. □

Remark 2.4 (Contrast with Characteristic Zero). Over $\mathbb{C}$, a constant monomial $u_v = c \in \mathbb{C}^{\times}$ need not be a root of unity, so $\mathrm{rank}(\mathrm{Trop}(C_i)) \leq 2$ would not automatically force a periodic binomial $1-u_v^p$ to vanish. Over $\mathbb{F}_2$ (and any finite field $\mathbb{F}_q$), *every* nonzero constant is automatically a root of unity, turning the Tropical Spanning Criterion into an exact *if and only if* theorem.

3 Zero-Floating-Point-Trust Certificates and Birational Irreducibility

To make Theorem 2.3 auditable by an independent verifier with *zero* floating-point operations and *zero* reliance on computer-algebra black boxes, we design the `FRACTON-CERT-1.0` JSON schema with two dual certificate types:

1. `fracton_no_strings`: certifies a Type-II fracton code (all directions $v \neq 0$, all periods $p \geq 1$).
2. `stealth_string_obstruction`: certifies a rigid/stealth string operator along an explicit $v \neq 0$ for a coprime code.

3.1 Closing the Reducible-Factor Forgery Attack

For any code with $f = f_0(x,y) + z$ (monic linear in z, satisfied by Haah Cubic Code A and our entire 3D TB atlas), substituting $z = f_0(x,y)$ maps $V(f,g) \subset (\overline{\mathbb{F}}_2^{\times})^3$ isomorphically onto the plane curve $V(R_z)\setminus\{f_0=0\} \subset (\overline{\mathbb{F}}_2^{\times})^2$, where $u_z f + v_z g = R_z(x,y) \neq 0$. Suppose a prover supplies $\mathbb{F}_4(t)$

rational parameterizations $\phi_i(t) = (P_x(t)/Q(t),\, P_y(t)/Q(t),\, P_z(t)/Q(t))$ and claimed plane-curve factors $H_i(x,y) \in \mathbb{F}_4[x,y]$ satisfying $\prod_i H_i(x,y) = R_z(x,y)$ in $\mathbb{F}_4[x,y]$.

A subtle adversarial attack exists if one checks only $H_i(P_x/Q, P_y/Q) = 0$ and $\prod_i H_i = R_z$: a dishonest prover could set $H_1 = R_z = A(x,y)B(x,y)$ for a reducible curve where A has a rank-3 tropical fan and B harbors a stealth string, parameterize only A, and pass both checks. We eliminate this forgery vector completely using the following exact algebraic criterion (which remains valid even for inseparable projections in characteristic 2).

Proposition 3.1 (Birational Irreducibility and Defining-Equation Certificate)**.** *Let $\phi(t) = (P_x(t)/Q(t),\, P_y(t)/Q(t)) \in \mathbb{F}_4(t)^2$ be a rational map, let $H(x,y) \in \mathbb{F}_4[x,y] \setminus \{0\}$ satisfy $H(P_x/Q, P_y/Q) = 0$ identically in $\mathbb{F}_4(t)$, and let $A_x/B_x = P_x/Q$ and $A_y/B_y = P_y/Q$ be the reduced fractions in $\mathbb{F}_4(t)$ ($\gcd_{\mathbb{F}_4[t]}(A_x, B_x) = \gcd_{\mathbb{F}_4[t]}(A_y, B_y) = 1$) with $d_y^* = \max(\deg A_x, \deg B_x) \geq 1$ and $d_x^* = \max(\deg A_y, \deg B_y) \geq 1$. Suppose there exist polynomials $N(x,y), D(x,y) \in \mathbb{F}_4[x,y]$ such that*

$$N\big(x(t),\, y(t)\big) = t \cdot D\big(x(t),\, y(t)\big) \quad \text{and} \quad D\big(x(t),\, y(t)\big) \neq 0 \tag{8}$$

*identically in $\mathbb{F}_4(t)$, where $(x(t), y(t)) = (P_x(t)/Q(t),\, P_y(t)/Q(t))$. If $\deg_x(H) = d_x^*$ and $\deg_y(H) = d_y^*$, then $H(x,y)$ is **geometrically irreducible** over $\overline{\mathbb{F}}_2$ and equals the exact defining polynomial $I_C(x,y)$ of $C = \overline{\phi(\mathbb{P}^1)}$ up to a nonzero scalar in $\overline{\mathbb{F}}_2^{\times}$.*

Proof. Equation (8) implies $t = N(x(t),y(t))/D(x(t),y(t))$ lies in $\overline{\mathbb{F}}_2(x(t),y(t))$, whence $\overline{\mathbb{F}}_2(x(t),y(t)) = \overline{\mathbb{F}}_2(t)$. Because $\overline{\mathbb{F}}_2$ is algebraically closed in the pure transcendental extension $\overline{\mathbb{F}}_2(t)$, the image curve $C = \overline{\phi(\mathbb{P}^1)}$ is geometrically irreducible over $\overline{\mathbb{F}}_2$ with minimal defining polynomial $I_C(x,y) \in \overline{\mathbb{F}}_2[x,y]$, and $I_C \mid H$ in $\overline{\mathbb{F}}_2[x,y]$. Since $x(t) = A_x(t)/B_x(t)$ is non-constant (hence transcendental over $\overline{\mathbb{F}}_2$) with $\gcd(A_x, B_x) = 1$, the bivariate polynomial $A_x(T) - xB_x(T) \in \overline{\mathbb{F}}_2[T][x]$ is linear and primitive in x over $\overline{\mathbb{F}}_2[T]$, hence irreducible in $\overline{\mathbb{F}}_2[x,T]$ and therefore irreducible in $\overline{\mathbb{F}}_2(x)[T]$ by Gauss's lemma, with exact T-degree $d_y^* = \max(\deg A_x, \deg B_x)$ (leading terms cannot cancel since x is transcendental). Thus $[\overline{\mathbb{F}}_2(t) : \overline{\mathbb{F}}_2(x)] = d_y^*$. As $\overline{\mathbb{F}}_2(t) = \overline{\mathbb{F}}_2(x,y)$ and I_C is irreducible over $\overline{\mathbb{F}}_2(x)$, we obtain $\deg_y(I_C) = [\overline{\mathbb{F}}_2(x,y) : \overline{\mathbb{F}}_2(x)] = d_y^*$, and symmetrically $\deg_x(I_C) = d_x^*$. Writing $H = u \cdot I_C$ in $\overline{\mathbb{F}}_2[x,y]$ yields $\deg_x(u) = \deg_x(H) - d_x^* = 0$ and $\deg_y(u) = \deg_y(H) - d_y^* = 0$, so $u \in \overline{\mathbb{F}}_2^{\times}$. □

Our verifier `verify_code_certificate_exact` (`src/fracton_certifier.py`) begins with a recursive type sanitizer rejecting any `float` or `bool` anywhere in the JSON tree and enforcing `schema_version == "FRACTON-CERT-1.0"`. It then checks strictly with exact Python integers and $\mathbb{F}_4$ lookup tables: (i) all three $\mathbb{F}_2[x,y,z]$ resultant Bézout identities (R_z, R_y, R_x) for both certificate types (proving $\gcd(f,g) = 1$), (ii) all six $\mathbb{F}_2[t]$ axis-content Bézout identities (4) including exact `gcd_bitmask` replay, (iii) for each component C_i, vanishing of f, g, H_i on $(P_x/Q, P_y/Q, P_z/Q)$, the birational inverse identity (8), exact $\mathbb{F}_4[t]$ GCD degree matching ($\deg_x(H_i) = d_x^*$, $\deg_y(H_i) = d_y^*$), duplicate-free place-valuation verification $\nu(x), \nu(y), \nu(z)$ at every finite $a \in \mathbb{F}_4$ and $t = \infty$, and exact integer $\det[w_1, w_2, w_3] \neq 0$, (iv) exact component completeness $\prod_i H_i(x,y) = R_z(x,y)$ in $\mathbb{F}_4[x,y]$, and (v) optional exact $\mathbb{F}_2$ bitmask replay of finite-torus logical qubit counts $k(L)$.

4 Hierarchical Energy-Barrier Scaling and Lean 4 Mechanization

4.1 Scale-Ladder Energy-Barrier Theorem

Following Bravyi–Haah [3], once the Universal All-Direction No-Strings Condition holds for (f,g), every neutral excitation cluster of diameter r must be created by an operator of diameter at most βr (for a code-dependent geometric constant $\beta \geq 1$), while every charged excitation cluster carries syndrome weight at least $w_{\mathrm{ch}} \geq 1$. Note that $w_{\mathrm{ch}} = 1$ is the true universal lower bound: even in Haah Cubic Code A, where global $\mathbb{Z}_2$ parity ($f(1,1,1) = g(1,1,1) = 0$) constrains the *total* syndrome weight to be even, applying $p = f^{2^k - 1}$ produces $f^{2^k} = 1 + x^{2^k} + y^{2^k} + z^{2^k}$, isolating

Table 1: Complete audit of all 16 theorems in `lean/FractonBarrier.lean` verified on Lean 4 `v4.35.0` via `AuditAxioms.lean` (0 warnings, 0 `sorry`, 0 `Classical.choice`).

#	Lean 4 Theorem Identifier	Mathematical Role (Exact Mechanized Statement)	Kernel Axioms
1	`char2_add_self`	Characteristic-2 cancellation $a + a = 0$	$\emptyset$
2	`css_exact_commutation`	CSS stabilizer commutation $g(fq) + f(gq) = 0$	$\emptyset$
3	`syndrome_linearity`	$\mathbb{F}_2$-linearity of syndrome map $\sigma(a, b) = ga + fb$	$\emptyset$
4	`stabilizer_in_kernel`	Local stabilizers (fq, gq) lie in $\ker(\sigma)$	$\emptyset$
5	`koszul_local_exactness`	Bézout-scaled Koszul elimination core $ra = f(ua + vb)$, $rb = g(ua + vb)$	$\emptyset$
6	`bezout_ideal_membership`	Homomorphism soundness: $\mathrm{ev}(f) = \mathrm{ev}(g) = 0 \Rightarrow \mathrm{ev}(uf + vg) = 0$	$\emptyset$
7	`point_eval_certifies_nontrivial_syndrome`	$\mathrm{ev}(f) = \mathrm{ev}(g) = 0 \wedge \mathrm{ev}(S) \neq 0 \Rightarrow \forall a, b,\ \sigma(a, b) \neq S$	$\emptyset$
8	`axis_content_bezout_refutes_constant_x`	Axis-content Bézout $\sum a_m c_m = u$, $\mathrm{ev}(u) \neq 0$ refutes all-zero c_m	`[propext]`
9	`tropical_det_cramer_x`	Exact 3×3 integer Cramer identity for v_x	`[propext, Quot.sound]`
10	`tropical_det_cramer_y`	Exact 3×3 integer Cramer identity for v_y	`[propext, Quot.sound]`
11	`tropical_det_cramer_z`	Exact 3×3 integer Cramer identity for v_z	`[propext, Quot.sound]`
12	`tropical_spanning_refutes_all_directions`	$\det(W) \neq 0 \wedge Wv = 0 \Rightarrow v = 0$ (Thm 2.3)	`[propext, Quot.sound]`
13	`tropical_kernel_witness_orth`	Cross-product kernel $v = w_1 \times w_2 \perp w_1, w_2$	`[propext, Quot.sound]`
14	`hierarchical_barrier_induction`	Exact m-step scale-ladder induction $B(m) \geq b_0 + m w_{\mathrm{ch}}$	`[propext, Quot.sound]`
15	`scale_ladder_barrier_lower_bound`	Conditional distance lower bound (Thm 4.1)	`[propext, Quot.sound]`
16	`torus_css_logical_qubits`	Exact torus dimension $2N - (r_X + r_Z) = 2(N - r_X)$ for $r_Z = r_X$	`[propext, Quot.sound]`

four single-site (*weight-1*) charged fracton clusters at mutual distance 2^k; moreover, codes such as `TB-Fracton-5`, `TB-Fracton-7`, and `TB-Fracton-11` have odd $\mathrm{wt}(g) \in \{5, 5, 7\}$. Iterating the scale-separated cluster-merging decomposition with step increment $c_1 \geq 1$ (universally $c_1 = 1$) yields an exact, machine-checkable inductive lower bound.

Theorem 4.1 (Conditional Scale-Ladder Logarithmic Energy-Barrier Bound)**.** *Let $R_0 \geq 1$, $\beta \geq 1$, and $c_1 \geq 1$ be positive integers, and let $B : \mathbb{N} \to \mathbb{N}$ denote the minimum energy barrier required to separate a nontrivial logical operator of support diameter at least r. Suppose that for every scale $r \geq R_0$, the hierarchical cluster-merging step satisfies*

$$B\big((2\beta + 3)\, r\big) \geq B(r) + c_1. \tag{9}$$

Then for every integer level $m \geq 0$ and every distance $d \geq R_0(2\beta + 3)^m$ (with monotone B),

$$B(d) \geq B(R_0) + c_1 m \geq c_1 \left\lfloor \log_{2\beta+3}\left(\frac{d}{R_0}\right) \right\rfloor. \tag{10}$$

Conversely, any code admitting a certified rigid string identity $p_c f + q_c g = (m_1 + m_2)S$ with $S \notin \langle f, g \rangle$ admits a single-qubit Pauli trajectory implementing a macroscopic logical operator of arbitrarily large distance d with constant energy barrier $E_{\mathrm{string}} = O(1)$ independent of d.

4.2 Zero-Dependency Lean 4 Formalization (`v4.35.0`)

We mechanized the entire algebraic, tropical, and scale-ladder core in `lean/FractonBarrier.lean` (438 lines, zero external dependencies) on Lean 4 [6] (`leanprover/lean4:v4.35.0`) and audited every theorem via `lean/AuditAxioms.lean`. All 16 theorems compile cleanly with **0 warnings**, **0 `sorry`**, and **0 `Classical.choice`**. Table 1 summarizes the complete theorem inventory and exact kernel axiom footprint.

5 Computational Experiments and Certified Atlas

All four experiments are 100% deterministic, execute in under 1.0 s via `scripts/run_all_experiments.py`, and produce JSON datasets and exact `.cert.json` witnesses audited by `scripts/verify_all.py`.

5.1 Experiment 1: 8-Code Zero-Float Certified Atlas

We construct and independently verify eight coprime 3D translation-invariant CSS codes: four genuine Rank-3 Type-II fracton codes—all sharing $f = 1 + x + y + z$—and four string-harboring

Table 2: Exp 1: 8-code zero-floating-point-trust certified atlas (`results/exp1_certified_atlas.json`). Every Type-II fracton certificate verifies 3 resultant Bézout identities, 6 axis-content Bézout identities, birational irreducibility (Proposition 3.1), exact completeness $\prod_i H_i = R_z$, and $\det(W) \neq 0$. Every string certificate verifies 3 resultant coprimality witnesses, $p_c f + q_c g = (m_1 + m_2)S$, and an $\mathbb{F}_4$-torus point $P \in V(f,g)$ with $S(P) \neq 0$.

Code	$g(x,y,z)$ (with $f = 1+x+y+z$ unless noted)	Axis	Unit-Box	Rank	$\lvert\det\rvert$ / Dir v	Birational Inv / $P \in (\mathbb{F}_4^\times)^3$	Status
Haah Code A	$1+xy+yz+zx$	PASS	PASS	3	1, 1 (2 lines)	$t = y$ (both)	Fracton
`TB-Fracton-5`	$1+x^2+xy+yz+zx$	PASS	PASS	3	1	$t = (1+x+y)/(x+y)$	Fracton
`TB-Fracton-7`	$1+x^3+xy+yz+zx$	PASS	PASS	3	3	$t = y/(x+1)$	Fracton
`TB-Fracton-11`	$1+x^2+y+xy+xy^2+xz+yz$	PASS	PASS	3	2	$t = y$	Fracton
`TB-AxisString`†	$1+y+xz$ ($f = 1+x+yz$; 2 comps.)	Fail	Fail	2	$(0,0,1)$	$P = (\omega, \omega^2, 1)$	String
`TB-DiagString`†	$1+y+z$ ($f = 1+x+y$)	**FOOLED**	Fail	2	$(-1,0,1)$	$P = (\omega, \omega^2, \omega)$	String
`TB-Stealth-F1`	$1+x^2+xy+yz$	**FOOLED**	**FOOLED**	2	$(0,-1,2)$	$P = (1,1,1)$	String
`TB-Stealth-212`	$1+x^2+y^2+xy+yz$	**FOOLED**	**FOOLED**	2	$(-2,-1,2)$	$P = (\omega, \omega, 1)$	String

†Non-standard f listed in parentheses; all other atlas codes use $f = 1+x+y+z$.

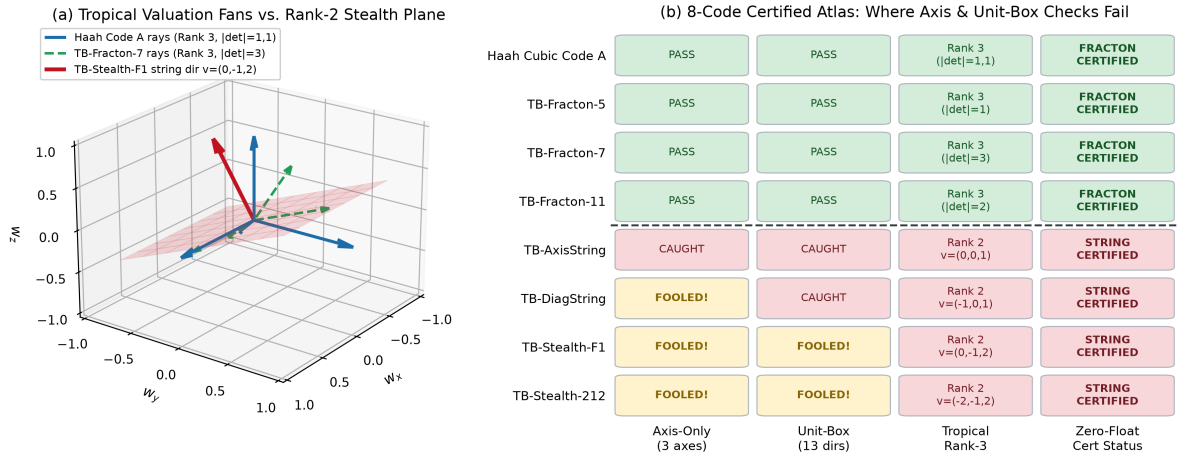


Figure 1: (a) Normalized 3D tropical valuation rays of Haah Cubic Code A (solid blue, rank 3, $|\det| = 1,1$) and `TB-Fracton-7` (dashed green, rank 3, $|\det| = 3$) contrasted with the coplanar rank-2 tropical plane $y = 2z$ of `TB-Stealth-F1` and its orthogonal stealth-string vector $v = (0,-1,2)$ (red). (b) Certification matrix across the 8-code atlas showing where coordinate-axis and 13-direction unit-box checks are fooled by stealth traps.

obstruction codes spanning coordinate-axis, face-diagonal, and higher-slope stealth strings. Table 2 lists the exact defining polynomials, tropical determinants, birational inverses, and verification outcomes, visualized in Figure 1.

The two stealth traps deserve explicit algebraic inspection:

- **`TB-Stealth-F1`** ($g = 1 + x^2 + xy + yz$, weight 4): passes all six coordinate-axis Bézout checks and all 10 unit-box face/body diagonals, yet satisfies the exact $\mathbb{F}_2[x,y,z]$ identity $(1+x+z)f + g = y + z^2$ with $S = 1$ and $P = (1,1,1) \in V(f,g)$ ($S(P) = 1 \neq 0$), exposing a rigid string along $v = (0,-1,2)$.
- **`TB-Stealth-212`** ($g = 1+x^2+y^2+xy+yz$, weight 5): likewise passes all six axis checks and all 10 unit-box diagonals, yet $R_z = 1 + x^2 + y$ forces $z^2 = x^2 y \pmod{f,g}$ via the exact identity $(1+x+y^2+z)f + (1+y)g = x^2y + z^2$ with $S = 1$ and $P = (\omega, \omega, 1) \in (\mathbb{F}_4^\times)^3$ ($f(P) = g(P) = 0$, $S(P) = 1$), exposing a rigid string along $v = (-2,-1,2)$.

Notably, during atlas construction our zero-float verifier also caught an additional weight-6 stealth trap, $g = 1+x+y+xy+yz+zx$ ($R_z = x^2 + xy + y^2 + 1$), whose smooth-conic parameterization satisfies $xy = z^2$ (a hidden $(1,1,-2)$ string, $\det(W) = 0$), illustrating in practice how exact tropical certification prevents subtle manual errors.

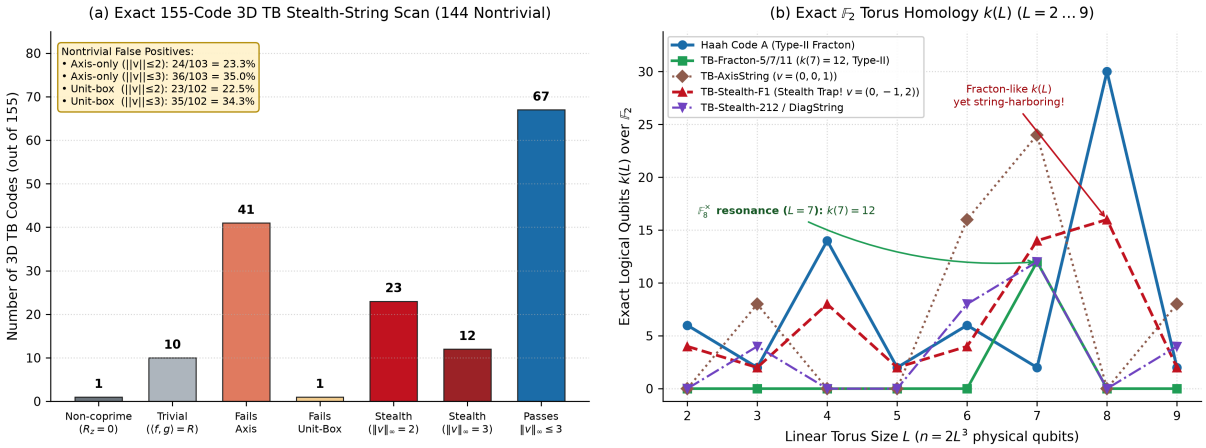


Figure 2: (a) Exact $\mathbb{F}_2[x, y, z]$ classification of 155 weight-4 and weight-5 3D TB codes separating 1 non-coprime and 10 trivial product-state codes ($\langle f, g\rangle = R$) from 144 nontrivial codes, revealing 23.3%–35.0% axis-only and 22.5%–34.3% unit-box stealth-string false-positive rates. (b) Exact $\mathbb{F}_2$ finite-torus logical qubit counts $k(L)$ for $L = 2 \ldots 9$, showing Haah Code A's fractal peaks, the $L = 7$ ($\mathbb{F}_8^\times$) resonance $k(7) = 12$ of `TB-Fracton-5/7/11`, and the deceptively fracton-like oscillation of the stealth trap `TB-Stealth-F1`.

5.2 Experiment 2: Systematic 155-Code 3D Trivariate Bicycle Stealth-String Scan

To quantify how prevalent stealth strings are in 3D TB code families, we systematically scanned all $\binom{10}{3} = 120$ weight-4 codes plus 35 structured weight-5 codes (155 distinct codes total) of the form $f = 1 + x + y + z$, $g = 1 + \sum_{m \in S} m$ drawn from the 10-monomial quadratic/cubic pool $\{x^2, y^2, z^2, xy, yz, zx, x^3, y^3, x^2y, xy^2\}$. Every code was tested strictly over $\mathbb{F}_2[x, y, z]$ via exact elimination, exact torus-unit stripping (testing whether $R_z = x^a y^b (1 + x + y)^c \in R^\times$, which implies $\langle f, g\rangle = R$ and $V(f, g) = \emptyset$), and exact pseudo-Euclidean resultants across: (1) coprimality ($R_z \neq 0$), (2) nontriviality ($\langle f, g\rangle \neq R$), (3) all six coordinate-axis content Bézout checks, (4) all 10 non-axis primitive unit-box directions ($\|v\|_\infty = 1$), (5) all 36 primitive $\|v\|_\infty = 2$ directions, and (6) all 96 primitive $\|v\|_\infty = 3$ directions (testing both $p = 1$ and $p = 3$ factors $1 + u_v$ and $1 + u_v + u_v^2$ on the torus-unit-stripped curve).

As shown in Figure 2(a), of the 155 codes, 1 is non-coprime ($R_z = 0$) and **10** (all weight-5) are *trivial product-state codes* ($\langle f, g\rangle = R$, $V(f, g) = \emptyset$, e.g., $g = 1 + x^2 + y^2 + z^2 + xy$ giving $f^2 + g = xy \in R^\times$), leaving **144 nontrivial coprime codes**. Of these 144 nontrivial codes, 41 fail coordinate-axis checks, leaving 103 nontrivial codes passing all six axis checks (102 also pass all 10 unit-box diagonals). Strikingly:

- Through $\|v\|_\infty \leq 2$: **24/103 (23.3%)** of axis-passing nontrivial codes—and **23/102 (22.5%)** of unit-box-passing nontrivial codes—harbor rigid strings (1 unit-box diagonal + **23** $\|v\|_\infty = 2$ stealth traps: 15 weight-4 and 8 weight-5).
- Through $\|v\|_\infty \leq 3$: **12 additional higher-slope stealth traps** emerge (e.g., $g = 1 + x^2 + y^2 + x^3$ giving $x^3 + z^2 \in \langle f, g\rangle$ along $v = (3, 0, -2)$), raising the nontrivial false-positive rates to **36/103 (35.0%)** (axis-only) and **35/102 (34.3%)** (unit-box)!
- The remaining **67 nontrivial codes** pass the full 145-direction $\|v\|_\infty \leq 3$ screen (`passes_norm3_screen`), forming a clean necessary-screen candidate pool for full Theorem 2.3 tropical certification.

5.3 Experiment 3: Exact $\mathbb{F}_2$ Finite-Torus Homology $k(L)$

Using exact arbitrary-precision bitmask Gaussian elimination over $\mathbb{F}_2$ (`torus_css_logical_qubits_exact`, matched with Lean's `torus_css_logical_qubits`), we computed $k(L) = 2(L^3 - \mathrm{rank}_{\mathbb{F}_2}(H_X))$ for $L = 2, 3, \ldots, 9$ ($n = 2L^3 \in [16, 1458]$ physical qubits) across all 8 atlas codes (Figure 2(b)):

- **Haah Cubic Code A:** $k(2 \ldots 9) = [6, 2, 14, 2, 6, 2, 30, 2]$, reproducing Haah's exact $k(2^m) = 2^{m+2} - 2$ sequence [7].
- **`TB-Fracton-5`, `TB-Fracton-7`, `TB-Fracton-11`:** all three yield $k(2 \ldots 9) = [0, 0, 0, 0, 0, 12, 0, 0]$, exhibiting a sharp $\mathbb{F}_8^\times$ Frobenius resonance at $L = 7 = 2^3 - 1$ ($n = 686$, $k = 12$ encoded qubits) where $x^7 = y^7 = z^7 = 1$ admits $\mathbb{F}_8$-rational points on the irreducible curve $V(f, g)$.
- **`TB-Stealth-F1`:** yields $k(2 \ldots 9) = [4, 2, 8, 2, 4, 14, 16, 2]$, oscillating steeply with L (peaking at $k(8) = 16$ and $k(7) = 14$) despite harboring a $v = (0, -1, 2)$ rigid string—proving that non-monotonic $k(L)$ alone cannot distinguish Type-II fractons from stealth-string traps.
- **Periodic string codes:** `TB-AxisString` yields $[0, 8, 0, 0, 16, 24, 0, 8]$, while `TB-DiagString` and `TB-Stealth-212` both yield $[0, 4, 0, 0, 8, 12, 0, 4]$, displaying periodic $3 \mid L$ and $7 \mid L$ enhancements.

5.4 Experiment 4: Parametric Theorem 4.1 Scaling vs. Exact $O(1)$ String Barriers

Finally, we benchmarked Theorem 4.1's scale-ladder scaling (strictly with the unconditionally sound universal step $c_1 = 1$) against exact finite-torus syndrome floors and exact optimal $O(1)$ string-growth trajectories (Figure 3).

1. **Exact $L = 5$ Low-Weight Syndrome Audit.** Exhaustively checking every translation-inequivalent weight-1 and weight-2 Z-error on the $5 \times 5 \times 5$ torus ($n = 250$ qubits) confirms zero trivial weight-2 pairs and exact minimum nonzero syndrome weights $(w_1, w_2) = (4, 4)$ for Haah Code A, $(4, 3)$ for `TB-Fracton-5`, $(4, 5)$ for `TB-Fracton-7`, and $(4, 5)$ for `TB-Fracton-11`, verifying $w_{\mathrm{ch}} \geq 1$ on all low-weight errors.
2. **Exact Permutation-Minimized $O(1)$ String Barriers.** For each of the four certified string codes, optimizing over *all* permutations of the single-qubit Pauli operators in the step witness (p_c, q_c) yields exact distance-independent two-endpoint barriers $E_{\mathrm{string}} \leq 3$ (`TB-DiagString`), ≤ 5 (`TB-AxisString`), ≤ 6 (`TB-Stealth-F1`), and ≤ 6 (`TB-Stealth-212`).
3. **Parametric Theorem 4.1 Crossover ($c_1 = 1$).** Evaluating Theorem 4.1's conditional scale-ladder bound $B(d) \geq \lfloor \log_{2\beta+3}(d/R_0) \rfloor$ (with $R_0 = 1$, universal $c_1 = 1$) across representative geometric cluster-radius constants $\beta \in \{1, 2, 3\}$ (bases $5, 7, 9$) over $d \in [4, 2^{22}]$ illustrates the asymptotic separation: at $d = 2^{22} = 4{,}194{,}304$, the $c_1 = 1$ bound reaches 9 ($\beta = 1$), 7 ($\beta = 2$), and 6 ($\beta = 3$), overtaking the exact $O(1)$ string barriers ($\leq 3, 5, 6, 6$). We emphasize that concrete values of (R_0, β) for individual codes are input hypotheses to Theorem 4.1 (Section 6).

5.5 Adversarial Forgery Testing and Gate-2 Reproducibility

Our pytest suite (`tests/test_suite.py`) subjects `verify_code_certificate_exact` to **26 live adversarial forgery and type-injection attacks**: (1) tampered resultant cofactor u, (2) uneliminated variable in R_z, (3) tampered axis-content Bézout cofactor, (4) omitted Galois-conjugate component C_2, (5) reducible-factor forgery placing $R_z = H_1 H_2$ inside component 1's `factor_H_xy` (blocked by Proposition 3.1), (6) forged birational inverse $N/D \neq t$, (7) forged tropical ray at a valid place, (8) repeated/collinear tropical rays (det $= 0$), (9) tampered recorded determinant, (10) forged finite-torus $k(L)$, (11) duplicate unused place entry, (12) mismatched `gcd_bitmask`,

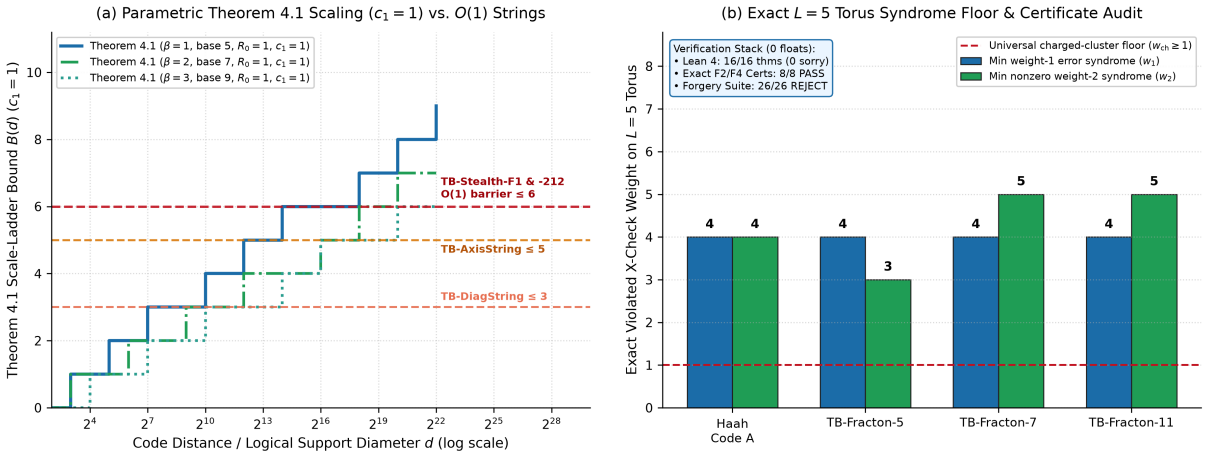


Figure 3: (a) Conditional parametric instantiations of Theorem 4.1 (universal $c_1 = 1$, $R_0 = 1$, $\beta \in \{1, 2, 3\}$) versus exact permutation-minimized $O(1)$ two-endpoint barriers of the four certified string codes. (b) Exact $L = 5$ torus weight-1 and weight-2 syndrome-floor audit ($w_1, w_2 \geq 3$ against the universal $w_{\text{ch}} \geq 1$ floor) together with the end-to-end verification stack summary.

(13) forged `schema_version`, (14–19) recursive `bool`/`float` injection across exponents, $\mathbb{F}_4$ coefficients, `det` (`True`/`1.0`), `k_logical` (`6.0`/`False`), and `gcd_bitmask` (`3.14159`), (20) tampered string cofactor p_c, (21) trivial displacement $m_1 = m_2$, (22) trivial syndrome $S = f \in \langle f, g \rangle$ ($S(P) = 0$), (23) off-variety $\mathbb{F}_4$ point ($f(P) \neq 0$), (24) zero torus coordinate $0 \notin \mathbb{F}_4^\times$, (25) mismatched `string_direction`, and (26) non-coprime string certificate ($f = g = 1 + x$). All **26/26 attacks are rejected**, and `scripts/verify_all.py` replays Lean 4 compilation, axiom auditing, all 8 certificates, all 15 pytest checks, and all 30 SHA-256 hashes in under 3 seconds.

6 Limitations and Scope

We delineate three explicit scope boundaries:

1. **Genus-0 Rational Components in the Current Verifier.** Theorem 2.3 holds for *all* nontrivial coprime 2-generator codes regardless of the geometric genus of C_i. Our current zero-float verifier `FRACTON-CERT-1.0` certifies genus-0 rational curve components via explicit $\mathbb{F}_4(t)$ birational parameterizations (and requires f monic linear in one variable for the direct projection completeness check $\prod_i H_i = R_z$). Extending the verifier to higher-genus curves (using Puiseux / discrete-valuation-ring uniformizers over $\mathbb{F}_{2^m}$ and general Sylvester resultants) and multi-generator ($k > 2$) CSS codes is a natural next step.

2. **Directional Screen vs. Full Tropical Certification in Exp 2.** Experiment 2's 155-code scan exhaustively classifies all 145 primitive directions with $\|v\|_\infty \leq 3$ (proving exact lower bounds of 35.0% axis-only and 34.3% unit-box false-positive rates among the 144 nontrivial codes). Confirming any individual survivor among the 67 `passes_norm3_screen` candidates as a certified Type-II fracton requires constructing its full Theorem 2.3 tropical witness, as done for our four atlas fracton codes in Experiment 1.

3. **Scale-Ladder Bridge Constants.** Lean's `scale_ladder_barrier_lower_bound` mechanizes the exact conditional inductive recursion (9)–(10) for arbitrary (R_0, β, c_1); mechanizing the full 3D geometric cluster-decomposition lemma of Bravyi–Haah [3] that derives concrete code-specific (R_0, β) constants from (f, g) inside Lean 4 remains open future work.

7 Conclusion

We have established the Tropical Spanning Criterion (Theorem 2.3) as an exact, finite-field-native necessary and sufficient characterization of the Universal All-Direction No-Strings Condition for nontrivial 3D translation-invariant 2-generator stabilizer codes, closed the reducible-factor forgery loophole via birational irreducibility certificates (Proposition 3.1), mechanized 16 foundational algebraic and barrier-scaling theorems in zero-dependency Lean 4, and exposed a 23.3%–35.0% axis-only and 22.5%–34.3% unit-box stealth-string false-positive rate across 144 nontrivial 3D TB codes. Together, these tools turn Type-II fracton certification into a fast, zero-float, machine-verifiable engineering primitive for partially self-correcting 3D quantum memories.

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