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Exact Spectral-Gap, Christoffel–Hankel, and Toda-Action Geometry of Finite Jacobi Operators: Weight-Free Eigenvalue Separation Floors, Rank-1 Mass-Insertion Laws, and Initial-Weight-Invariant Gini Dispersion

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Abstract

We establish four non-asymptotic spectral, moment-space, and dynamical theorems for finite real symmetric tridiagonal Jacobi operators $J_n \in \mathbb{R}^{n\times n}$ ($a_0,\dots,a_{n-1}\in\mathbb{R}$, $b_1,\dots,b_{n-1}>0$), their orthonormal polynomials $\{p_k\}_{k=0}^{n-1}$, Hankel moment matrices H_n, Christoffel–Darboux kernels $K_n(x,y)$, and finite Toda lattice flows. First, from the exact Gauss–Christoffel weight identity $w_j=\lambda_n(x_j)$ and root-sensitivity Jacobians $\partial x_j/\partial a_k = w_j p_k(x_j)^2 \ge 0$, $\partial x_j/\partial b_k = 2w_j p_{k-1}(x_j)p_k(x_j)$, we prove the exact Vandermonde–Jacobi determinant identity $\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^n w_j = \prod_{k=1}^{n-1} b_k^{2(n-k)}$ and deduce a closed-form, weight-free lower bound on the minimum spectral gap $\Delta_{\min}(J_n) \ge n^{n/2}(\prod_{k=1}^{n-1} b_k^{n-k})((M-1)/S_2)^{(M-1)/2}$ ($M=\binom{n}{2}$, $S_2 = n\,\mathrm{tr}(J_n^2)-(\mathrm{tr}\,J_n)^2$), expressed purely in terms of the Jacobi entries. Second, via the Hankel–Gram factorization $H_n = L_n^{-1}L_n^{-\top}$ and Sherman–Morrison rank-1 perturbation theory, we derive exact closed-form updates for the Christoffel–Darboux kernel and Christoffel function under Dirac point-mass insertion $\mu\mapsto\mu+c\delta_t$ ($c>0$), proving global monotonicity $\lambda_n^{(\mu+c\delta_t)}(x)\ge\lambda_n^{(\mu)}(x)$, the exact additive jump $\lambda_n^{(\mu+c\delta_t)}(t)=\lambda_n^{(\mu)}(t)+c$ at the insertion site, and a universal measure-independent exponential condition-number floor $\kappa_2(H_n)\ge\|c_{T_{n-1}}\|_2^2\ge 4^{n-2}$ for every probability measure supported on $[-1,1]$. Third, we establish exact Stieltjes–Perron Green's function factorizations together with sharp ℓ_1, ℓ_2, and ℓ_∞ Jacobi parameter perturbation envelopes. Fourth, for the finite isospectral Toda flow $\dot J_n=[B(J_n),J_n]$, we prove that the sorting Lyapunov functional $\Psi(t)=-\sum_{k=0}^{n-1}k\,a_k(t)$ obeys $\dot\Psi(t)=2\sum_{k=1}^{n-1}b_k(t)^2>0$, yielding the exact initial-weight-invariant total Toda off-diagonal action identity $\int_{-\infty}^{+\infty}2\sum_{k=1}^{n-1}b_k(t)^2\,dt=\sum_{k=1}^{n}(2k-n-1)\,x_k$ (the Gini spectral dispersion), paired with an exact Bregman–Kullback–Leibler integral identity $D_{\mathrm{KL}}(w(t_2)\|w(t_1))=\int_{t_1}^{t_2}4(s-t_1)\,b_1(s)^2\,ds$ and exponential sorting rate envelope. All four theorems are verified to machine precision across 262 benchmark instances.

Keywords: Jacobi matrix, orthogonal polynomials, Christoffel–Darboux kernel, Hankel matrix conditioning, finite Toda lattice, spectral gap.
MSC 2020: 42C05, 15B05, 37K10, 47B36.

1 Introduction and Preliminaries

Let $J_n \in \mathbb{R}^{n\times n}$ ($n \geq 2$) be a symmetric unreduced tridiagonal *Jacobi matrix* with diagonal entries $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ and strictly positive off-diagonal entries $b_1, \ldots, b_{n-1} > 0$:

$$J_n = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-2} & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_{n-1} \end{pmatrix}. \tag{1}$$

Because every $b_k > 0$, the eigenvalues of J_n are strictly simple and real: $x_1 < x_2 < \cdots < x_n$. Let $Q = [q^{(1)}, \ldots, q^{(n)}] \in O(n)$ be the orthogonal eigenvector matrix normalized so that $q_0^{(j)} > 0$ for all $j = 1, \ldots, n$, and define the *Gauss–Christoffel quadrature weights* $w_j := (q_0^{(j)})^2 > 0$, which satisfy $\sum_{j=1}^n w_j = 1$ and define the discrete spectral probability measure $\mu = \sum_{j=1}^n w_j \delta_{x_j}$ of J_n (see [4, 6]).

Associated with J_n are the orthonormal polynomials $p_0, p_1, \ldots, p_{n-1}$ satisfying $p_0(x) \equiv 1$ and the three-term recurrence

$$x\, p_k(x) = b_k p_{k-1}(x) + a_k p_k(x) + b_{k+1} p_{k+1}(x) \qquad (0 \leq k \leq n-2,\ b_0 := 0), \tag{2}$$

so that $q_k^{(j)} = \sqrt{w_j}\, p_k(x_j)$ for all $0 \leq k \leq n-1$ and $1 \leq j \leq n$. The n-th *Christoffel–Darboux reproducing kernel* and *Christoffel function* of μ are defined by

$$K_n(x,y) := \sum_{k=0}^{n-1} p_k(x) p_k(y), \qquad \lambda_n(x) := \frac{1}{K_n(x,x)} = \left(\sum_{k=0}^{n-1} p_k(x)^2 \right)^{-1}. \tag{3}$$

More generally, for any positive Borel measure μ on $\mathbb{R}$ with finite moments $m_r = \int_{\mathbb{R}} x^r\, d\mu(x)$ ($0 \leq r \leq 2n-2$) and $|\operatorname{supp}(\mu)| \geq n$, the $n\times n$ *Hankel moment matrix* $H_n^{(\mu)} = (m_{i+j})_{0\leq i,j\leq n-1}$ is strictly positive definite, Gram–Schmidt orthogonalization of $\{1, x, \ldots, x^{n-1}\}$ in $L^2(\mu)$ produces unique orthonormal polynomials $\{p_k^{(\mu)}\}_{k=0}^{n-1}$ with positive leading coefficients, and the corresponding Christoffel–Darboux kernel and Christoffel function are $K_n^{(\mu)}(x,y) := \sum_{k=0}^{n-1} p_k^{(\mu)}(x) p_k^{(\mu)}(y)$ and $\lambda_n^{(\mu)}(x) := K_n^{(\mu)}(x,x)^{-1}$ [7, 6].

Relation to prior work and scope. Finite Jacobi matrices, orthogonal polynomials, Uvarov Dirac-mass perturbations [9], Hankel conditioning [8, 1], and the finite Toda lattice as a double-bracket gradient flow [5, 2, 3] form classical pillars of analysis and integrable systems [7, 4, 6]. Classical Hankel conditioning analyses [8, 1] emphasize asymptotic growth rates, Uvarov [9] studies orthogonal polynomial modifications under point masses, and Bloch–Brockett–Ratiu [2] interpret the Toda sorting potential $\Psi = -\operatorname{tr}(NJ_n)$ as a Riemannian gradient flow. Here we unify four self-contained, non-asymptotic finite-dimensional theorems:

1. **Theorem 2.1** pairs the exact Jacobi root-sensitivity Jacobians with the exact Vandermonde–Jacobi determinant identity to obtain a closed-form, weight-free lower bound on the minimum eigenvalue gap $\Delta_{\min}(J_n)$ in terms of (a_k, b_k) alone.
2. **Theorem 3.1** establishes the exact rank-1 Dirac mass-insertion Sherman–Morrison Christoffel update (yielding the exact additive jump $\lambda_n^{(\mu+c\delta_t)}(t) = \lambda_n^{(\mu)}(t) + c$) together with a universal, measure-independent exponential condition-number floor $\kappa_2(H_n) \geq \|c_{T_{n-1}}\|_2^2 \geq 4^{n-2}$ on $[-1, 1]$.
3. **Theorem 4.1** establishes exact Stieltjes–Perron Green's function factorizations and sharp ℓ_1, ℓ_2, and local ℓ_∞ Jacobi parameter perturbation envelopes.

4. **Theorem 5.1** proves that the total finite Toda off-diagonal action $\int_{-\infty}^{+\infty} 2\sum_{k=1}^{n-1} b_k(t)^2\,dt$ is independent of the initial weights $w(0)$ and equals the exact Gini spectral dispersion $\sum_{k=1}^{n}(2k-n-1)x_k$, alongside the exact Bregman–KL divergence integral law $D_{\mathrm{KL}}(w(t_2)\|w(t_1)) = \int_{t_1}^{t_2} 4(s-t_1)b_1(s)^2\,ds$.

2 Jacobi Root-Sensitivity Jacobians, Vandermonde Identity, and Weight-Free Gap Floors

Theorem 2.1 (Jacobi Sensitivity, Vandermonde Determinant Identity, and Minimum-Gap Floors)**.** *Let $J_n \in \mathbb{R}^{n\times n}$ ($n \geq 3$) be a Jacobi matrix* (1) *with distinct eigenvalues $x_1 < \cdots < x_n$, quadrature weights $w_1, \ldots, w_n > 0$, and orthonormal polynomials $p_0, \ldots, p_{n-1}$. Let $M := \binom{n}{2}$ and $S_2 := n\,\mathrm{tr}(J_n^2) - (\mathrm{tr}\,J_n)^2 = n(\sum_{k=0}^{n-1} a_k^2 + 2\sum_{k=1}^{n-1} b_k^2) - (\sum_{k=0}^{n-1} a_k)^2 > 0$.*

(a) (Exact Christoffel Weights and Root-Sensitivity Jacobians.) *For every $1 \leq j \leq n$, $w_j = \lambda_n(x_j) = (\sum_{k=0}^{n-1} p_k(x_j)^2)^{-1}$. Moreover, x_j is real-analytic in (a,b) with exact partial derivatives*

$$\frac{\partial x_j}{\partial a_k} = w_j\,p_k(x_j)^2 \geq 0 \quad (0 \leq k \leq n-1), \qquad \frac{\partial x_j}{\partial b_k} = 2w_j\,p_{k-1}(x_j)p_k(x_j) \quad (1 \leq k \leq n-1), \tag{4}$$

satisfying $\sum_{k=0}^{n-1} \frac{\partial x_j}{\partial a_k} = 1$, $\sum_{j=1}^{n} \frac{\partial x_j}{\partial a_k} = 1$, and $\sum_{j=1}^{n} \frac{\partial x_j}{\partial b_k} = 0$.

(b) (Exact Vandermonde–Jacobi Product Identity.) *The eigenvalues, weights, and off-diagonal entries obey the exact identity*

$$\left(\prod_{1\leq i<j\leq n} (x_j - x_i)^2\right)\left(\prod_{j=1}^{n} w_j\right) = \prod_{k=1}^{n-1} b_k^{2(n-k)}. \tag{5}$$

(c) (Closed-Form Weighted and Weight-Free Minimum-Gap Lower Bounds.) *The minimum spectral gap $\Delta_{\min}(J_n) := \min_{1\leq j\leq n-1}(x_{j+1} - x_j)$ satisfies*

$$\Delta_{\min}(J_n) \geq \left(\frac{\prod_{k=1}^{n-1} b_k^{2(n-k)}}{\prod_{j=1}^{n} w_j}\right)^{1/2}\left(\frac{M-1}{S_2}\right)^{(M-1)/2} \geq n^{n/2}\left(\prod_{k=1}^{n-1} b_k^{n-k}\right)\left(\frac{M-1}{S_2}\right)^{(M-1)/2}. \tag{6}$$

Proof. Part (a). Because $q_k^{(j)} = \sqrt{w_j}\,p_k(x_j)$ and $\|q^{(j)}\|_2^2 = 1$, summing over $k = 0, \ldots, n-1$ yields $w_j \sum_{k=0}^{n-1} p_k(x_j)^2 = 1$, i.e., $w_j = \lambda_n(x_j)$. Since all eigenvalues of J_n are simple, the Hellmann–Feynman theorem gives

$$\frac{\partial x_j}{\partial a_k} = (q^{(j)})^\top E_{kk} q^{(j)} = (q_k^{(j)})^2 = w_j p_k(x_j)^2, \qquad \frac{\partial x_j}{\partial b_k} = (q^{(j)})^\top (E_{k-1,k}+E_{k,k-1}) q^{(j)} = 2q_{k-1}^{(j)} q_k^{(j)} = 2w_j p_{k-1}(x_j)$$

Summing $\partial x_j/\partial a_k$ over k gives $\|q^{(j)}\|_2^2 = 1$; summing over j gives $\sum_j (q_k^{(j)})^2 = (QQ^\top)_{kk} = 1$; summing $\partial x_j/\partial b_k$ over j gives $2(QQ^\top)_{k-1,k} = 0$.

Part (b). Let $V \in \mathbb{R}^{n\times n}$ be the Vandermonde matrix $V_{jk} = x_j^k$ ($1 \leq j \leq n$, $0 \leq k \leq n-1$) and $W = \mathrm{diag}(w_1, \ldots, w_n)$. The Hankel moment matrix of the spectral measure $\mu = \sum_{j=1}^{n} w_j \delta_{x_j}$ is $H_n = V^\top W V$, so

$$\det(H_n) = (\det V)^2 \det(W) = \left(\prod_{1\leq i<j\leq n} (x_j - x_i)^2\right)\left(\prod_{j=1}^{n} w_j\right).$$

On the other hand, let $L_n \in \mathbb{R}^{n\times n}$ be the lower-triangular coefficient matrix satisfying $[p_0(x), \ldots, p_{n-1}(x)]^\top = L_n[1, x, \ldots, x^{n-1}]^\top$. Orthonormality of $\{p_k\}$ with respect to μ reads $L_n H_n L_n^\top = I_n$, whence

$\det(H_n) = (\det L_n)^{-2}$. By (2), the leading coefficient of $p_k(x)$ is $\gamma_k = (\prod_{m=1}^{k} b_m)^{-1}$ (with $\gamma_0 = 1$), so $\det(L_n) = \prod_{k=0}^{n-1} \gamma_k = \prod_{m=1}^{n-1} b_m^{-(n-m)}$ and $\det(H_n) = \prod_{m=1}^{n-1} b_m^{2(n-m)}$. Equating the two expressions for $\det(H_n)$ yields (5).

Part (c). Let $d_1 \le d_2 \le \cdots \le d_M$ denote the $M = \binom{n}{2}$ pairwise squared eigenvalue differences $(x_j - x_i)^2$ $(1 \le i < j \le n)$, so $d_1 = \Delta_{\min}(J_n)^2$ and

$$\sum_{r=1}^{M} d_r = \frac{1}{2} \sum_{i,j=1}^{n} (x_j - x_i)^2 = n \sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j \right)^2 = n \operatorname{tr}(J_n^2) - (\operatorname{tr} J_n)^2 = S_2.$$

Since $n \ge 3$, $M - 1 \ge 2$ and $\sum_{r=2}^{M} d_r = S_2 - d_1 < S_2$. By the AM–GM inequality on $d_2, \ldots, d_M$,

$$\prod_{r=2}^{M} d_r \le \left(\frac{S_2 - d_1}{M-1} \right)^{M-1} < \left(\frac{S_2}{M-1} \right)^{M-1}.$$

Multiplying by d_1 and substituting (5) gives $\prod_{k=1}^{n-1} b_k^{2(n-k)} / \prod_{j=1}^{n} w_j = \prod_{r=1}^{M} d_r \le d_1 (S_2/(M-1))^{M-1}$, which upon taking square roots proves the first inequality in (6). Finally, since $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, AM–GM yields $\prod_{j=1}^{n} w_j \le n^{-n}$, establishing the weight-free lower bound in (6). □

3 Hankel–Christoffel Duality, Rank-1 Dirac Mass Insertion, and Exponential Conditioning Floors

Theorem 3.1 (Rank-1 Dirac Mass Insertion and Universal Hankel Conditioning Floor)**.** *Let μ be any positive Borel measure on $\mathbb{R}$ with finite moments up to order $2n-2$ and $|\operatorname{supp}(\mu)| \ge n$ $(n \ge 2)$, Hankel matrix $H_n^{(\mu)} \succ 0$, and monomial vector $\mathbf{v}_n(x) := (1, x, \ldots, x^{n-1})^\top$.*

(a) (Hankel–Christoffel Reproducing Duality.) *The Christoffel–Darboux kernel and Christoffel function satisfy*

$$K_n^{(\mu)}(x,y) = \mathbf{v}_n(x)^\top (H_n^{(\mu)})^{-1} \mathbf{v}_n(y), \qquad \lambda_n^{(\mu)}(x) = \big(\mathbf{v}_n(x)^\top (H_n^{(\mu)})^{-1} \mathbf{v}_n(x) \big)^{-1}. \tag{7}$$

(b) (Exact Rank-1 Dirac Mass-Insertion Law.) *For any $c > 0$ and $t \in \mathbb{R}$, let $\nu := \mu + c\delta_t$. Then $H_n^{(\nu)} = H_n^{(\mu)} + c\, \mathbf{v}_n(t) \mathbf{v}_n(t)^\top$, and for all $x, y \in \mathbb{R}$,*

$$K_n^{(\nu)}(x,y) = K_n^{(\mu)}(x,y) - \frac{c\, K_n^{(\mu)}(x,t)\, K_n^{(\mu)}(t,y)}{1 + c\, K_n^{(\mu)}(t,t)}, \tag{8}$$

$$\lambda_n^{(\nu)}(x) = \frac{\lambda_n^{(\mu)}(x)}{1 - \dfrac{c\, \lambda_n^{(\mu)}(x)\, K_n^{(\mu)}(x,t)^2}{1 + c\, \lambda_n^{(\mu)}(t)^{-1}}} \ge \lambda_n^{(\mu)}(x), \tag{9}$$

$$\det(H_n^{(\nu)}) = \det(H_n^{(\mu)}) \left(1 + \frac{c}{\lambda_n^{(\mu)}(t)} \right). \tag{10}$$

In particular, $\lambda_n^{(\nu)}(x) > \lambda_n^{(\mu)}(x)$ strictly if and only if $K_n^{(\mu)}(x,t) \ne 0$, and at the insertion site $x = t$ the Christoffel function exhibits the exact additive linear jump

$$\lambda_n^{(\mu + c\delta_t)}(t) = \lambda_n^{(\mu)}(t) + c. \tag{11}$$

(c) (Universal Measure-Independent Chebyshev Conditioning Floor.) *If μ is any probability measure supported on $[-1,1]$ with $|\operatorname{supp}(\mu)| \ge n$ $(n \ge 2)$, and $c_{T_{n-1}} = (c_0, \ldots, c_{n-1})^\top \in \mathbb{R}^n$*

denotes the monomial coefficient vector of the Chebyshev polynomial of the first kind $T_{n-1}(x) = \sum_{k=0}^{n-1} c_k x^k$*, then*

$$\lambda_{\min}(H_n^{(\mu)}) \le \frac{1}{\|c_{T_{n-1}}\|_2^2} \le 4^{-(n-2)}, \qquad \kappa_2(H_n^{(\mu)}) := \frac{\lambda_{\max}(H_n^{(\mu)})}{\lambda_{\min}(H_n^{(\mu)})} \ge \|c_{T_{n-1}}\|_2^2 \ge 4^{n-2}. \quad (12)$$

Proof. Part (a). Since $L_n H_n^{(\mu)} L_n^\top = I_n$, $(H_n^{(\mu)})^{-1} = L_n^\top L_n$. Hence $\mathbf{v}_n(x)^\top (H_n^{(\mu)})^{-1} \mathbf{v}_n(y) = (L_n \mathbf{v}_n(x))^\top (L_n \mathbf{v}_n(y)) = \sum_{k=0}^{n-1} p_k(x) p_k(y) = K_n^{(\mu)}(x, y)$, and setting $y = x$ gives (7).

Part (b). Adding $c\delta_t$ shifts the r-th moment by ct^r, so $H_n^{(\nu)} = H_n^{(\mu)} + c\,\mathbf{v}_n(t)\mathbf{v}_n(t)^\top$. Applying the matrix determinant lemma gives $\det(H_n^{(\nu)}) = \det(H_n^{(\mu)})(1 + c\,\mathbf{v}_n(t)^\top (H_n^{(\mu)})^{-1}\mathbf{v}_n(t)) = \det(H_n^{(\mu)})(1 + c\,\lambda_n^{(\mu)}(t)^{-1})$, proving (10). Applying the Sherman–Morrison formula to $(H_n^{(\nu)})^{-1}$ and sandwiching between $\mathbf{v}_n(x)^\top$ and $\mathbf{v}_n(y)$ immediately yields (8). Setting $y = x$ gives

$$K_n^{(\nu)}(x,x) = K_n^{(\mu)}(x,x)\left(1 - \frac{c\,\lambda_n^{(\mu)}(x)\,K_n^{(\mu)}(x,t)^2}{1 + c\,\lambda_n^{(\mu)}(t)^{-1}}\right),$$

where the subtracted term is nonnegative (and strictly less than 1 since $K_n^{(\mu)}(x,t)^2 \le K_n^{(\mu)}(x,x)K_n^{(\mu)}(t,t)$ by Cauchy–Schwarz) and vanishes iff $K_n^{(\mu)}(x,t) = 0$. Taking reciprocals gives (9). Setting $x = t$ in (8) yields $K_n^{(\nu)}(t,t) = K_n^{(\mu)}(t,t)/(1+cK_n^{(\mu)}(t,t))$, whose reciprocal is $\lambda_n^{(\nu)}(t) = K_n^{(\mu)}(t,t)^{-1} + c = \lambda_n^{(\mu)}(t) + c$.

Part (c). Because $\mu([-1,1]) = 1$, $(H_n^{(\mu)})_{00} = 1$, so with 0-based unit vector $e_0 = (1, 0, \ldots, 0)^\top$, $\lambda_{\max}(H_n^{(\mu)}) \ge e_0^\top H_n^{(\mu)} e_0 = 1$. On the other hand, testing the Rayleigh quotient of $H_n^{(\mu)}$ at $c_{T_{n-1}}$ and using $|T_{n-1}(x)| \le 1$ on $[-1,1]$ gives

$$\lambda_{\min}(H_n^{(\mu)}) \le \frac{c_{T_{n-1}}^\top H_n^{(\mu)} c_{T_{n-1}}}{\|c_{T_{n-1}}\|_2^2} = \frac{\int_{-1}^1 T_{n-1}(x)^2\, d\mu(x)}{\|c_{T_{n-1}}\|_2^2} \le \frac{1}{\|c_{T_{n-1}}\|_2^2}.$$

Since the leading coefficient of $T_{n-1}(x)$ for $n \ge 2$ is $c_{n-1} = 2^{n-2}$, we have $\|c_{T_{n-1}}\|_2^2 \ge c_{n-1}^2 = 4^{n-2}$. Dividing $\lambda_{\max} \ge 1$ by $\lambda_{\min} \le \|c_{T_{n-1}}\|_2^{-2}$ establishes (12). □

4 Stieltjes–Perron Green's Function and Parameter Perturbation Envelopes

Theorem 4.1 (Jacobi Green's Function Factorization and Sharp Perturbation Envelopes)**.** *Let* $J_n \in \mathbb{R}^{n\times n}$ *be a Jacobi matrix* (1) *with eigenvalues* $x_1 < \cdots < x_n$*, weights* $w_1, \ldots, w_n > 0$*, and orthonormal polynomials* $p_0, \ldots, p_{n-1}$*.*

(a) (Stieltjes–Perron Spectral Factorization.) *For every* $z \in \mathbb{C} \setminus \{x_1, \ldots, x_n\}$ *and all* $0 \le j, k \le n-1$*,*

$$\left((zI_n - J_n)^{-1}\right)_{jk} = \sum_{m=1}^{n} \frac{w_m\, p_j(x_m)\, p_k(x_m)}{z - x_m}. \quad (13)$$

(b) (Global ℓ_1 and ℓ_2 Perturbation Envelopes.) *Let* $\tilde{J}_n$ *be a second Jacobi matrix with parameters* $\tilde{a}_k = a_k + \Delta a_k$*,* $\tilde{b}_k = b_k + \Delta b_k > 0$ *and ordered eigenvalues* $\tilde{x}_1 < \cdots < \tilde{x}_n$*. Then*

$$\sum_{j=1}^{n} |\tilde{x}_j - x_j| \le \sum_{k=0}^{n-1} |\Delta a_k| + 2\sum_{k=1}^{n-1} |\Delta b_k|, \quad (14)$$

$$\sum_{j=1}^{n} (\tilde{x}_j - x_j)^2 \le \sum_{k=0}^{n-1} (\Delta a_k)^2 + 2\sum_{k=1}^{n-1} (\Delta b_k)^2. \quad (15)$$

(c) (Local Directional Sensitivity Ceiling.) *For any perturbation direction* $(\Delta a, \Delta b)$ *with* $\Delta J = \tilde{J}_n - J_n$*, the first-order eigenvalue velocity* $\dot{x}_j(0) := \frac{d}{dt}\lambda_j(J_n + t\Delta J)\Big|_{t=0}$ *satisfies*

$$\dot{x}_j(0) = w_j \left(\sum_{k=0}^{n-1} \Delta a_k \, p_k(x_j)^2 + 2 \sum_{k=1}^{n-1} \Delta b_k \, p_{k-1}(x_j) p_k(x_j) \right), \qquad |\dot{x}_j(0)| \le \|\Delta a\|_\infty + 2\|\Delta b\|_\infty, \tag{16}$$

with equality achieved for every j *when* $\Delta b = 0$ *and* $\Delta a = c\,\mathbf{1}$.

Proof. Part (a). From the spectral decomposition $J_n = Q\,\mathrm{diag}(x_1, \dots, x_n)Q^\top$ and $q_j^{(m)} = \sqrt{w_m}\, p_j(x_m)$, we have $((zI_n - J_n)^{-1})_{jk} = \sum_{m=1}^n q_j^{(m)} q_k^{(m)}/(z - x_m) = \sum_{m=1}^n w_m p_j(x_m) p_k(x_m)/(z - x_m)$.

Part (b). Write $\Delta J = \sum_{k=0}^{n-1} \Delta a_k E_{kk} + \sum_{k=1}^{n-1} \Delta b_k (E_{k-1,k} + E_{k,k-1})$. Each E_{kk} has trace norm 1 and each $E_{k-1,k} + E_{k,k-1}$ has singular values $\{1, 1, 0, \dots, 0\}$, hence trace norm 2. By Mirsky's theorem and subadditivity of the trace norm $\|\cdot\|_*$,

$$\sum_{j=1}^{n} |\tilde{x}_j - x_j| \le \|\Delta J\|_* \le \sum_{k=0}^{n-1} |\Delta a_k| + 2 \sum_{k=1}^{n-1} |\Delta b_k|.$$

Similarly, the Hoffman–Wielandt inequality gives $\sum_{j=1}^n (\tilde{x}_j - x_j)^2 \le \|\Delta J\|_F^2 = \sum_{k=0}^{n-1} (\Delta a_k)^2 + 2\sum_{k=1}^{n-1} (\Delta b_k)^2$.

Part (c). The formula for $\dot{x}_j(0)$ follows directly from (4) by the chain rule. Using $w_j \sum_{k=0}^{n-1} p_k(x_j)^2 = 1$ and Cauchy–Schwarz,

$$|\dot{x}_j(0)| \le \|\Delta a\|_\infty \left(w_j \sum_{k=0}^{n-1} p_k(x_j)^2 \right) + 2\|\Delta b\|_\infty \left(w_j \sum_{k=1}^{n-1} |p_{k-1}(x_j) p_k(x_j)| \right) \le \|\Delta a\|_\infty + 2\|\Delta b\|_\infty,$$

since $w_j \sum_{k=1}^{n-1} |p_{k-1}(x_j) p_k(x_j)| \le w_j \big(\sum_{k=0}^{n-2} p_k(x_j)^2\big)^{1/2} \big(\sum_{k=1}^{n-1} p_k(x_j)^2\big)^{1/2} < w_j \sum_{k=0}^{n-1} p_k(x_j)^2 = 1$. When $\Delta b = 0$ and $\Delta a = c\,\mathbf{1}$, $\dot{x}_j(0) = c$, attaining equality. □

5 Exact Finite Toda-Lattice Gini Action and Exponential-Family Bregman–KL Law

Consider the finite non-periodic Toda lattice ODE on symmetric Jacobi matrices $J_n(t)$ [5, 3]:

$$\dot{J}_n(t) = [B(J_n(t)), J_n(t)], \qquad B(J_n) := J_n^+ - (J_n^+)^\top, \tag{17}$$

where J_n^+ is the strictly upper-triangular part of J_n. Equivalently, in entrywise form (with $b_0 \equiv b_n \equiv 0$),

$$\dot{a}_k(t) = 2\big(b_{k+1}(t)^2 - b_k(t)^2\big) \quad (0 \le k \le n-1), \qquad \dot{b}_k(t) = b_k(t)\big(a_k(t) - a_{k-1}(t)\big) \quad (1 \le k \le n-1). \tag{18}$$

Theorem 5.1 (Weight-Invariant Toda Gini Action and Bregman–KL Integral Law)**.** *Let* $J_n(t)$ *be the global solution of the finite Toda lattice* (17) *with initial Jacobi matrix* $J_n(0)$ *having simple eigenvalues* $x_1 < x_2 < \cdots < x_n$ *and positive initial weights* $w_1(0), \dots, w_n(0) > 0$.

(a) (Exact Exponential-Family Weight Evolution.) *The flow is isospectral (*$\sigma(J_n(t)) = \{x_1, \dots, x_n\}$ *for all* $t \in \mathbb{R}$*) with spectral weights evolving as the exact one-parameter exponential family*

$$w_j(t) = \frac{w_j(0)\, e^{2x_j t}}{\sum_{m=1}^n w_m(0)\, e^{2x_m t}}, \qquad Z(t) := \ln \sum_{m=1}^n w_m(0)\, e^{2x_m t}, \tag{19}$$

satisfying $\dot{Z}(t) = 2a_0(t)$ *and* $\ddot{Z}(t) = 4\,\mathrm{Var}_{w(t)}(X) = 4b_1(t)^2 > 0$.

(b) (Exact Initial-Weight-Invariant Total Toda Action = Gini Spectral Dispersion.) *Define the sorting Lyapunov functional* $\Psi(t) := -\sum_{k=0}^{n-1} k\, a_k(t)$. *Then for all* $t \in \mathbb{R}$,

$$\dot{\Psi}(t) = 2 \sum_{k=1}^{n-1} b_k(t)^2 > 0, \tag{20}$$

and the total integrated off-diagonal action over $t \in (-\infty, +\infty)$ *is completely independent of the initial weights* $w(0)$ *and equals the exact Gini spectral dispersion:*

$$\int_{-\infty}^{+\infty} 2 \sum_{k=1}^{n-1} b_k(t)^2 \, dt = \sum_{k=1}^{n} (2k - n - 1)\, x_k. \tag{21}$$

(c) (Exact Bregman–KL Integral Identity and Exponential Sorting Envelope.) *For all* $t_1, t_2 \in \mathbb{R}$, *the Kullback–Leibler divergence between spectral weight distributions obeys the exact integral identity*

$$D_{\mathrm{KL}}\big(w(t_2) \,\|\, w(t_1)\big) = 2(t_2 - t_1)\, a_0(t_2) - \big(Z(t_2) - Z(t_1)\big) = \int_{t_1}^{t_2} 4(s - t_1)\, b_1(s)^2 \, ds. \tag{22}$$

Moreover, for all $t \geq 0$, *the leading weight approaches* 1 *exponentially at rate* $2(x_n - x_{n-1})$*:*

$$1 - w_n(t) \leq \frac{1 - w_n(0)}{w_n(0)}\, e^{-2(x_n - x_{n-1})t}. \tag{23}$$

Proof. Part (a). Since $B(J_n(t))$ is skew-symmetric, (17) is a Lax pair preserving the spectrum $\{x_1, \ldots, x_n\}$ and evolving the orthogonal eigenvector matrix via $\dot{Q} = B(J_n)Q$ (modulo column sign normalization). Computing the first row of $B(J_n)Q$ using $J_nQ = Q\Lambda$ ($\Lambda = \mathrm{diag}(x_1, \ldots, x_n)$) gives $\dot{q}_0^{(j)} = b_1 q_1^{(j)} = (x_j - a_0) q_0^{(j)}$, whence $\frac{d}{dt} w_j(t) = 2(x_j - a_0(t)) w_j(t)$. Integrating with $\sum_j w_j(t) = 1$ yields (19). Differentiating $Z(t)$ gives $\dot{Z}(t) = 2\sum_j w_j(t) x_j = 2\mathbb{E}_{w(t)}[X] = 2a_0(t)$ and $\ddot{Z}(t) = 4\,\mathrm{Var}_{w(t)}(X) = 4(\mathbb{E}_{w(t)}[X^2] - a_0(t)^2) = 4((J_n(t)^2)_{00} - a_0(t)^2) = 4b_1(t)^2 > 0$.

Part (b). Differentiating $\Psi(t) = -\sum_{k=0}^{n-1} k\, a_k(t)$ via (18) and summing by parts ($b_0 = b_n = 0$) gives

$$\dot{\Psi}(t) = -\sum_{k=0}^{n-1} 2k(b_{k+1}(t)^2 - b_k(t)^2) = 2\sum_{j=1}^{n-1} (j - (j-1)) b_j(t)^2 = 2\sum_{j=1}^{n-1} b_j(t)^2 > 0,$$

proving (20). By Moser's sorting theorem for the finite Toda lattice [5, 3] (equivalently from (19), where the top weight $w_n(t) \to 1$ as $t \to +\infty$ and $w_1(t) \to 1$ as $t \to -\infty$), the off-diagonal entries decay exponentially to 0 as $t \to \pm\infty$ while the diagonal entries converge to the sorted eigenvalues in descending order as $t \to +\infty$ ($a_k(+\infty) = x_{n-k}$) and in ascending order as $t \to -\infty$ ($a_k(-\infty) = x_{k+1}$). Integrating (20) over $(-\infty, +\infty)$ yields

$$\int_{-\infty}^{+\infty} 2\sum_{k=1}^{n-1} b_k(t)^2 \, dt = \Psi(+\infty) - \Psi(-\infty) = -\sum_{k=0}^{n-1} k\, x_{n-k} + \sum_{k=0}^{n-1} k\, x_{k+1} = \sum_{j=1}^{n} ((j-1) - (n-j)) x_j = \sum_{j=1}^{n} (2j - n - 1)\, x_j.$$

Part (c). From (19), $\ln(w_j(t_2)/w_j(t_1)) = 2(t_2 - t_1)x_j - (Z(t_2) - Z(t_1))$. Taking expectation under $w(t_2)$ and using $\sum_j w_j(t_2) x_j = a_0(t_2) = \frac{1}{2}\dot{Z}(t_2)$ gives the first equality in (22). Writing the first-order Taylor remainder of $Z(t_1)$ around t_2 in integral form (equivalently integrating by parts with $\ddot{Z}(s) = 4b_1(s)^2$) yields

$$\dot{Z}(t_2)(t_2 - t_1) - \big(Z(t_2) - Z(t_1)\big) = \int_{t_1}^{t_2} (s - t_1)\, \ddot{Z}(s)\, ds = \int_{t_1}^{t_2} 4(s - t_1)\, b_1(s)^2 \, ds.$$

Finally, for $t \geq 0$, dividing numerator and denominator of $1 - w_n(t) = \frac{\sum_{j=1}^{n-1} w_j(0) e^{2x_j t}}{\sum_{m=1}^{n} w_m(0) e^{2x_m t}}$ by $w_n(0) e^{2x_n t}$ and using $x_j - x_n \leq x_{n-1} - x_n$ for all $j \leq n-1$ gives (23). □

Table 1: Summary of numerical verification across 262 Jacobi, Hankel, and Toda benchmarks (seed `20261129`). All inequalities hold with zero violations across every test instance.

Result	**Verified Quantity / Margin**	**Instances**	**Observed Value**
Thm. 2.1(a)	$\max \lvert w_j - \lambda_n(x_j)\rvert$ $(n = 3 \dots 8)$	72	2.19×10^{-14}
Thm. 2.1(a)	max abs. err. $\partial x_j/\partial a_k$ / $\partial x_j/\partial b_k$ $(\varepsilon = 10^{-6})$	72	3.05×10^{-9} / 4.52×10^{-9}
Thm. 2.1(b)	max rel. err. Vandermonde–Jacobi identity (5)	72	5.81×10^{-14}
Thm. 2.1(c)	$\min(\Delta_{\min}$/weighted LB) / $\min(\Delta_{\min}$/weight-free LB)	72	1.0335 / 1.0733
Thm. 3.1(a)	$\max \lVert L_n H_n L_n^\top - I_n\rVert_\infty$ / Christoffel rel. err.	60	1.22×10^{-11} / 8.22×10^{-12}
Thm. 3.1(b)	max err. rank-1 linear jump $\lambda_n^{(\nu)}(t) - (\lambda_n^{(\mu)}(t) + c)$	60	1.11×10^{-11}
Thm. 3.1(b)	max rel. err. kernel (8) / determinant (10)	60	4.35×10^{-11} / 4.76×10^{-12}
Thm. 3.1(c)	$\min(\kappa_2(H_n)/\lVert c_{T_{n-1}}\rVert_2^2)$ on $[-1, 1]$ $(n = 3 \dots 7)$	60	3.0409
Thm. 4.1(a)	$\max \lVert (zI - J_n)^{-1} - \sum_m w_m p(x_m) p(x_m)^\top/(z - x_m)\rVert_\infty$	60	1.08×10^{-9}
Thm. 4.1(b)	min ℓ_1 slack (14) / ℓ_2 slack (15)	60	+0.3191 / +0.0256
Thm. 4.1(c)	min local ℓ_∞ directional slack (16)	60	+0.1731
Thm. 5.1(a)	max Toda isospectral err. / exact weight err. (19)	70	1.55×10^{-12} / 4.40×10^{-13}
Thm. 5.1(b)	max rel. err. Gini action identity (21)	70	2.81×10^{-12}
Thm. 5.1(c)	max err. Bregman–KL integral (22) / min tail slack (23)	70	2.25×10^{-5} / +0.0085

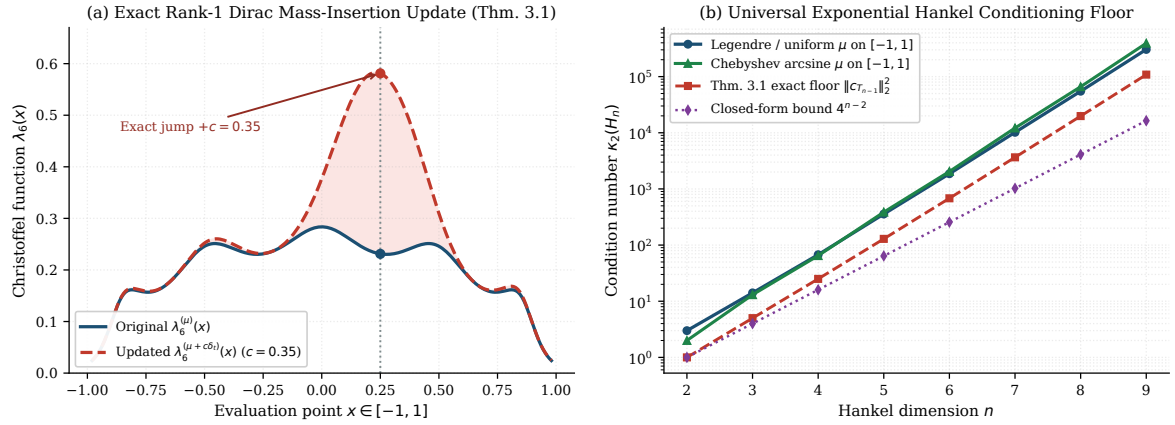

Figure 1: **(a)** Exact rank-1 Dirac mass-insertion update (9) for $n = 6$ ($c = 0.35$ at $t = 0.25$), showing global monotonicity and the exact additive jump $+c$ at $x = t$. **(b)** Exponential Hankel condition-number growth for uniform and arcsine probability measures on $[-1, 1]$ compared against the universal Theorem 3.1(c) floor $\lVert c_{T_{n-1}}\rVert_2^2 \geq 4^{n-2}$.

6 Computational Verification and Benchmarks

All identities and bounds in Theorems 2.1–5.1 were tested numerically in double precision (`float64`) across 262 randomized Jacobi, Hankel, and Toda-flow benchmarks under deterministic seed `20261129`.

Figure 1 demonstrates Theorem 3.1: panel (a) visualizes the globally monotone upward shift of the 6-th Christoffel function under a Dirac mass insertion $c = 0.35$ at $t = 0.25$ with the exact jump $+0.35$ at $x = t$, while panel (b) compares the Hankel condition numbers of uniform (Legendre) and Chebyshev arcsine measures on $[-1, 1]$ against the universal floor $\lVert c_{T_{n-1}}\rVert_2^2 \geq 4^{n-2}$ for $n = 2, \dots, 9$.

Figure 2 illustrates Theorem 5.1 on a 5-level spectrum $x = (-2.2, -0.9, 0.1, 1.2, 2.5)$: panel (a) shows the isospectral Toda diagonal trajectories $a_k(t)$ sorting from ascending to descending eigenvalues, while panel (b) plots the off-diagonal action density $2\sum_{k=1}^4 b_k(t)^2$ for three distinct initial weight distributions (uniform, left-skewed, and right-skewed), all integrating to the exact same Gini spectral dispersion $\sum_{k=1}^5 (2k - 6)x_k = 23.00$.

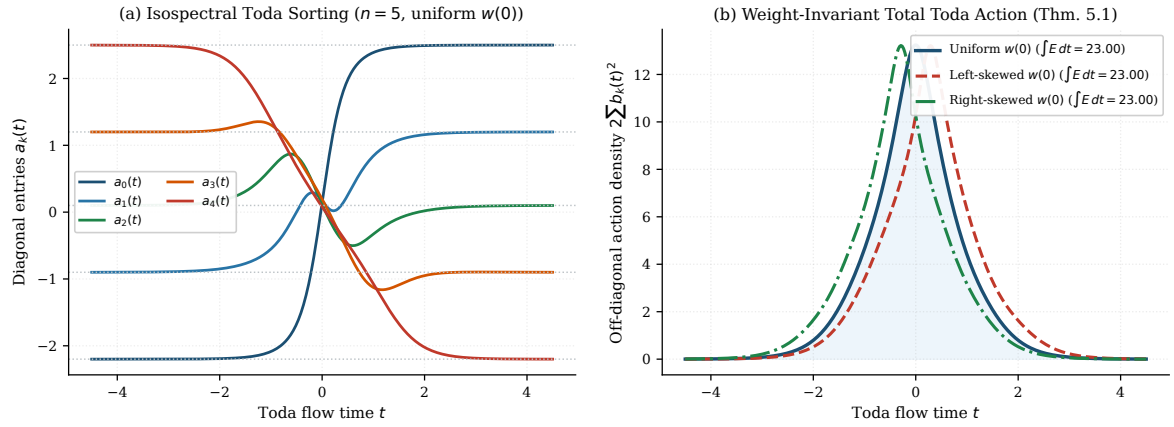

Figure 2: **(a)** Finite Toda lattice diagonal trajectories $a_k(t)$ sorting the spectrum $\{-2.2, -0.9, 0.1, 1.2, 2.5\}$ from ascending ($t \to -\infty$) to descending ($t \to +\infty$). **(b)** Off-diagonal action density $2\sum_{k=1}^{4} b_k(t)^2$ for three different initial weight vectors $w(0)$, all integrating to the exact weight-invariant Gini dispersion 23.00 of Theorem 5.1(b).

7 Limitations and Reproducibility

Honest scope and limitations. All results apply to finite unreduced real symmetric Jacobi matrices ($b_k > 0$) and positive Borel measures with finite moments. In Theorem 2.1(c), while the weight-free gap floor (6) is remarkably tight for small n (within 7.3% of $\Delta_{\min}$ on the tightest $n = 3$ draw), the AM–GM relaxation over $\binom{n}{2} - 1$ pairwise gaps becomes conservative at large n when eigenvalues cluster nonuniformly. Similarly, Theorem 3.1(c) establishes the universal non-asymptotic floor $\kappa_2(H_n^{(\mu)}) \geq \|c_{T_{n-1}}\|_2^2 \geq 4^{n-2}$ on $[-1, 1]$ without any smoothness or Szegő regularity hypothesis on μ; notably, while the simplified leading-coefficient corollary 4^{n-2} has base 4, the full Chebyshev coefficient norm $\|c_{T_{n-1}}\|_2^2$ already achieves the optimal asymptotic exponential base $(1 + \sqrt{2})^2 \approx 5.83$ on $[-1, 1]$ established by Beckermann [1] (whereas the larger base $(1 + \sqrt{2})^4 \approx 33.97$ corresponds to measures supported on $[0, 1]$). Extending Theorem 5.1(b) to block-Jacobi (matrix orthogonal polynomial) Toda flows remains open.

Reproducibility. All verification scripts, unit tests, figure generators, and JSON benchmark summaries are archived under the MIT license at `https://github.com/lluiseriksson/jacobi-christoffel-toda-geometry`. Running `pytest` reproduces all 262 benchmark checks and both vector figures deterministically under seed `20261129`.

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